Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72

CBA 4. Including Uncertainty

George E. Apostolakis
Massachusetts Institute of Technology

Spring 2007
Uncertainty

• Practically any CBA requires consideration of uncertainty.

• Most methodologies in use are *ad hoc*, due to the intrinsic difficulty of the generalized problem.
Methods

1. Scenario analysis
2. Adjustments of interest rates
3. Decision Theory
4. Simplified probabilistic models
Scenario Analysis

• Preparation and analysis of scenarios:
  -- “Optimistic” or “most favorable estimate”
  -- “Most likely” or “best estimate” or “fair estimate”
  -- “Pessimistic” or “least favorable estimate”

• Interpretation is difficult without assignment of probabilities to scenarios.

• Benefit: Brings additional information into the process.
Example

• A new machine is to be purchased for producing units in a new manner.

<table>
<thead>
<tr>
<th></th>
<th>Pessim.</th>
<th>Fair</th>
<th>Optim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual number of units:</td>
<td>900</td>
<td>1,000</td>
<td>1,100</td>
</tr>
<tr>
<td>Savings per unit:</td>
<td>$50</td>
<td>55</td>
<td>60</td>
</tr>
<tr>
<td>Operating costs:</td>
<td>$2,000</td>
<td>1,600</td>
<td>1,200</td>
</tr>
<tr>
<td>etc.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PW:</td>
<td>-$49,000</td>
<td>22,000</td>
<td>120,000</td>
</tr>
</tbody>
</table>

• What are we to do with such information?
Developing Scenarios

• For each element in the problem, e.g., interest rate and costs, define the three values.
• We really don’t know how conservative (pessimistic) the final answer is.
• People are bad processors of information.
• Point estimates tend to cluster around the median value. Possibility of displacement bias.
• Extremes greater than the 75th or smaller than the 25th percentile are difficult to imagine.
• Overconfidence.
Forecasting Oil Prices


• In 1980, 43 economists and energy experts forecast the price of oil from 1981 to 2020 to aid in policy planning.

• They used 10 leading econometric models under each of 12 scenarios embodying a variety of assumptions about inputs, such as supply, demand, and growth rates.
The Plausible Scenario

• One scenario was termed as the “plausible median case.” It represented “the general trends to be expected.”

• The 10 models were applied to the plausible scenario.

• Results for 1986:
  - Actual price: $13
  - Range of predictions: $27 to $51.
High Interest Rates

• Justify choice of alternatives using a high interest rate, e.g., 30%.

• **Example**

Total annual income: $55,000
Capital cost: $80,000
Annual capital recovery with return: $80,000 (A/P, 30%, 6yrs) = $30,272
Annual operating cost $28,600
Net annual profit: 55,000 – (30,272 + 28,600) = -$3,872
Example (cont’d)

- The high rate of 30% is intended to cover uncertainty.
- If the annual income were $60,000, then the net annual profit would be 60,000 - (30,272 + 28,600) = $1,128 and the venture would be accepted.
- A high interest rate does not guarantee that all uncertainties are accounted for. Its choice is arbitrary.
Decision Theory: Manufacturing Example

- **Decision:** To continue producing old product (O) or convert to a new product (N).

  The payoffs depend on the market conditions:

  - s: strong market for the new product
  - w: weak market for the new product
Manufacturing Example Payoffs

• **Earnings (payoffs):**

  L₁: $15,000/yr, old product,
  L₂: $30,000/yr, new product and the market is strong,
  L₃: -$10,000/yr, new product and the market is weak

• **Demand and Probabilities:**

<table>
<thead>
<tr>
<th>Period 1 (5 yrs)</th>
<th>Period 2 (5 yrs)</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>s₁</td>
<td>s₂</td>
<td>0.4 = P(s₁s₂)</td>
</tr>
<tr>
<td>s₁</td>
<td>w₂</td>
<td>0.4 = P(s₁w₂)</td>
</tr>
<tr>
<td>w₁</td>
<td>w₂</td>
<td>0.2 = P(w₁w₂)</td>
</tr>
</tbody>
</table>

P(s₁) = P(s₁s₂) + P(s₁w₂) = 0.8; P(w₁) = 0.2; P(s₂/s₁) = 0.5; P(w₂/w₁) = 1.0
Decision Tree

Decision Options

States of Nature

P1

P2

$243.26K

$96.94K

-$81.09K

$121.61K

PW of Payoffs

CBA 4. Including Uncertainty
Calculation of the Payoffs

\[
\text{PW}_{s1} = 30x(P / A, 0.04, 5) = 30x \frac{(1 + 0.04)^5 - 1}{0.04x(1 + 0.04)^5} = 133.52K
\]

\[
\text{PW}_{s2} = 30x(P / A, 0.04, 5)x(P / F, 0.04, 5) = \\
= 30x \frac{(1 + 0.04)^5 - 1}{0.04x(1 + 0.04)^5} \times \frac{1}{(1 + 0.04)^5} = 109.74K
\]

\[
\text{PW}_{s1s2} = 133.52 + 109.74 = 243.26K
\]

\[
\text{PW}_{w1} = -133.52x \frac{10}{30} = -44.51K \\
\text{PW}_{w2} = -109.74x \frac{10}{30} = -36.58K
\]

\[
\text{PW}_{s1w2} = 133.52 - 36.58 = 96.94K
\]

\[
\text{PW}_{w1w2} = -44.51 - 36.58 = -81.09K
\]
Calculation of the EMV

**Old Product**

\[ PW_O = 15x(P/A, 0.04, 10) = 15x \frac{(1 + 0.04)^{10} - 1}{0.04x(1 + 0.04)^{10}} = 121.61K \]

**New Product**

\[ EMV_N = 243.26x0.4 + 96.94x0.4 - 81.09x0.2 = 119.86K \]

**Decision**

Stay with the old product?
Calculation using Utilities

Let the utility of payoffs be \( U(x) = 1.18 \ln(x+5) - 1.29 \)
\(-2 \leq x \leq 2 \) (x in $M) \[U(2) = 1, U(-2)= 0\]
\( U(243.26) = 0.665; \ U(121.61) = 0.637; \ U(96.94) = 0.632; \)
\( U(-81.09) = 0.590 \)

Old Product
EU(O) = 0.637

New Product
EU(N) = 0.665\times0.4 + 0.632\times0.4 + 0.590\times0.2 = 0.6368

Decision
Stay with the old product?
Probabilistic Models

• We have:

\[ PW[X(T)] = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + \ldots + \frac{X_T}{(1+i)^T} \]  

(1)

where \( X_j \equiv B_j - C_j \) are the net benefits in year \( j \).

• All \( X_j \) are r.v.’s, \( \Rightarrow \) \( PW[X(T)] \) is a r.v.

• Note that (1) is of the form:

\[ Y = a_0 X_0 + a_1 X_1 + a_2 X_2 + \ldots + a_T X_T \]
Computing the probability density function (pdf) of PW is usually difficult in practice.

Try to compute the quantities $E[PW]$, the expected value of PW,

and $\sigma_{PW}^2$ i.e., the variance of PW.
• Let \( Z = aW + b \)

\( Z \) and \( W \) are r.v.’s, \( a \) and \( b \) constants

\[ \Rightarrow \quad E[Z] = aE[W] + b \]

\[ \Rightarrow \quad \sigma_Z^2 = a^2 \sigma_W^2 \]
Fundamental Relationships from Probability Theory (2)

- Let

\[ Z = W_1 + W_2 \quad (W_1, W_2 : \text{independent r.v.'s}) \]

\[ \Rightarrow \quad E[Z] = E[W_1] + E[W_2] \]

\[ \Rightarrow \quad \sigma^2_Z = \sigma^2_{W_1} + \sigma^2_{W_2} \]

**Note:** Extends to any number of mutually independent r.v.’s.
Fundamental Relationships from Probability Theory (3)

- Let

\[ Z = aW_1 + bW_2 + c \]

\[ \Rightarrow \]

\[ E[Z] = aE[W_1] + bE[W_2] + c \]

\[ \sigma_Z^2 = a^2 \sigma_{W_1}^2 + b^2 \sigma_{W_2}^2 \]

- If \( W_1 \) and \( W_2 \) are normal, then \( Z \) is also normal.

- We often assume that \( Z \) is normal even if \( W_1 \) and \( W_2 \) are not.
Example: Reliability Physics

• (RPRA 3, slide 30) A capacitor is placed across a power source. Assume that surge voltages occur on the line at a rate of one per month and they are normally distributed with a mean value of 100 volts and a standard deviation of 15 volts. The breakdown voltage of the capacitor is 130 volts.

• Suppose that the breakdown voltage is also normally distributed with standard deviation of 15 volts.
Example (2)

- The capacitor fails when the surge voltage, $S$, is greater than the capacity, $C$.
- $S$: rv with $E[S] = 100$, $\sigma_S = 15$ volts
- $C$: rv with $E[C] = 130$, $\sigma_C = 15$ volts
- Define a new rv $D \equiv C - S = aC + bS$
- Then, $E[D] = 130 - 100 = 30$ volts

and

$$\sigma_D = \sqrt{\sigma_C^2 + \sigma_S^2} = 21.21 \text{ volts}$$
Example (3)

- D is also normally distributed, therefore

\[ P_{d/sv}(D < 0) = P(Z < -\frac{30}{21.21}) = P(Z < -1.41) = \]
\[ P(Z > 1.41) = 0.5 - 0.42 = 0.08 \]

- RPRA 3, page 31, shows that \( P_{d/sv} = \) conditional probability of damage given a surge voltage
\[ = P(\text{surge voltage} > 130 \text{ volts/surge voltage}) \]
\[ = P(Z > \frac{130 - 100}{15}) = P(Z > 2) = \]
\[ = 1 - P(Z < 2) = 1 - 0.9772 = 0.0228 < 0.08 \]

The uncertainty in the breakdown voltage increased the failure probability.
Assuming Independence of $X_j$

$\text{PW}_{\text{Ind}}[X(T)] = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + \ldots + \frac{X_T}{(1+i)^T}$

- $X_j \ (j = 0, 1, \ldots, T)$ are mutually independent r.v.’s with known $E[X_j]$ and $\sigma^2_{X_j} = \sigma^2_j$

$E[\text{PW}_{\text{Ind}}\{X(T)\}] = E[Y_{\text{Ind}}] = \sum_{j=0}^{T} \frac{E[X_j]}{(1+i)^j}$

$\sigma^2_{\text{PW}_{\text{Ind}}} = \sigma^2_{Y_{\text{Ind}}} = \sum_{j=0}^{T} \frac{\sigma^2_j}{(1+i)^{2j}}$

- **Note:** Gaussian approximation for pdf of $Y$ may work well in this case.
**Example: Project A**

**T = 3 yrs.; i = 8%; Initial Cost = $10K**

<table>
<thead>
<tr>
<th>Probability, p</th>
<th>t=1</th>
<th>t=2</th>
<th>t=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>$3K</td>
<td>$3K</td>
<td>$3K</td>
</tr>
<tr>
<td>0.25</td>
<td>$4K</td>
<td>$4K</td>
<td>$4K</td>
</tr>
<tr>
<td>0.30</td>
<td>$5K</td>
<td>$5K</td>
<td>$5K</td>
</tr>
<tr>
<td>0.25</td>
<td>$6K</td>
<td>$6K</td>
<td>$6K</td>
</tr>
<tr>
<td>0.10</td>
<td>$7K</td>
<td>$7K</td>
<td>$7K</td>
</tr>
<tr>
<td><strong>1.00</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example (2)

- Denote with $X_1$, $X_2$, and $X_3$, the net benefits of A in years 1, 2 and 3, respectively.
- **Note:** Net benefits, $X_1$, $X_2$, and $X_3$, do not have to be identically distributed or symmetric or discrete; these choices are made just to keep the example simple.
- Denote with $Y$ the present worth of A, $PW(A)$. Then:

$$PW_{\text{Ind}}(A) = Y_{\text{Ind}} = -10K + \frac{X_1}{(1.08)} + \frac{X_2}{(1.08)^2} + \frac{X_3}{(1.08)^3}$$
Observations

- Y is a *random variable* (takes more than one value with different probabilities for any given implementation of project A)

- Value of Y will be determined by the values of the combination of X1, X2, and X3 that will actually materialize

- Corresponding *a priori* probability of any value of Y is equal to probability of that particular combination of X1, X2 and X3.
Expectation and Variance of Annual Net Benefits

- It is easy to determine the expected value and variance of each of $X_1$, $X_2$, and $X_3$, separately:

$$E[X_1] = 0.1x_3 + 0.25x_4 + 0.3x_5 + 0.25x_6 + 0.1x_7 = 5,000$$

(Similarly, we have $E[X_2] = 5K$ and $E[X_3] = 5K$.)

$$
\sigma_{X_1}^2 = 0.1(3-5)^2 + 0.25(4-5)^2 + 0.3(5-5)^2 + 0.25(6-5)^2 + \\
+ 0.1(7-5)^2 = 1,300,000 \approx (1,140)^2
$$

or, $\sigma_{X_1} \approx 1,140 = \sigma_{X_2} = \sigma_{X_3}$
Independence: Calculations

• Assume that the net benefits obtained from Project A in years 1, 2 and 3 are determined independently of one another.
• This means the probability of the combination \( \{X_1 = 3, X_2 = 6, X_3 = 4\} \) is equal to

\[
P(X_1 = 3, X_2 = 6, X_3 = 4) = P(X_1 = 3) \cdot P(X_2 = 6) \cdot P(X_3 = 4) = (0.1)(0.25)(0.25) = 0.00625
\]

that is, with probability 0.00625, the r.v. \( Y \), i.e., the PW of Project A, will take on the value

\[
PW_{\text{Ind}} = Y_{\text{Ind}} = -10K + 3K/(1.08) + 6K/(1.08)^2 + 4K/(1.08)^3 \approx 1,097.14
\]
Independence: Calculations (2)

- Note that \( Y \) can take on a total of 125 (= 5·5·5) different values of independent outcomes in years 1, 2 and 3.
- Using the expressions on slide 21 we get

\[
E[Y_{\text{Ind}}] = -10K + E[X1]/(1.08) + E[X2]/(1.08)^2 + E[X3]/(1.08)^3 = \\
= -10K + (5K)[1/1.08 + 1/(1.08)^2 + 1/(1.08)^3] \\
= -10K + (5K)\cdot(P/A, 0.08, 3) \approx -10K + (5K)(2.5771) \approx $2,885
\]

\[
\sigma_{Y,\text{Ind}}^2 = \sigma_{X1}^2 \cdot [1/(1.08)^2] + \sigma_{X2}^2 \cdot [1/(1.08)^4] + \sigma_{X3}^2 \cdot [1/(1.08)^6] \\
= (1,140)^2 \cdot [1/(1.08)^2 + 1/(1.08)^4 + 1/(1.08)^6] = \\
= (1,140)^2 \cdot (2.2225), \text{ or, } \sigma_{Y,\text{Ind}} = (1,140)\cdot(2.2225)^{1/2} \approx $1,700
\]
Conclusion

• Project A will, "on average," have a net present value equal to about $2,885 and a standard deviation of approximately $1,700 around that average.
What can we do with this information?

<table>
<thead>
<tr>
<th>Alternative</th>
<th>$E[Y] = E[PW]$</th>
<th>$\sigma_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>
Possible Decision Criteria

• Choose the alternative with the highest mean value (A1).
• Note how close A4 is and how large the standard deviations are. Depending on the uncertainties, A1 and A4 may be indistinguishable.

• Minimize the probability of loss.
• Assume normal distributions and find P(PW < 0).
Probability of Loss

- A1: \[ P(PW < 0) = P(Z < -(20/15)) = P(Z < -1.33) = 0.09 \] “best” alternative
- A2: \[ P(PW < 0) = P(Z < -(5/7)) = P(Z < -0.71) = 0.24 \]
- A3: \[ P(PW < 0) = P(Z < -(15/12)) = P(Z < -1.25) = 0.10 \]
- A4: \[ P(PW < 0) = P(Z < -(17/16)) = P(Z < -1.06) = 0.14 \]
Probability of Loss for Project A

- $P_{\text{Ind}}(PW_{\text{Ind}} < 0) = P(Z < -(2885/1700)) = P(Z < -1.7) = 0.04$

- The fundamental assumption is that of independence of the annual benefits.
Complete Dependence of $X_j$

- Once the net benefits, $X_1$, for year 1 are known, we shall also know exactly the net benefits for years 2 and 3.

<table>
<thead>
<tr>
<th>Probability, $p$</th>
<th>$t=1$</th>
<th>$t=2$</th>
<th>$t=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>$3K$</td>
<td>$3K$</td>
<td>$3K$</td>
</tr>
<tr>
<td>0.25</td>
<td>$4K$</td>
<td>$4K$</td>
<td>$4K$</td>
</tr>
<tr>
<td>0.30</td>
<td>$5K$</td>
<td>$5K$</td>
<td>$5K$</td>
</tr>
<tr>
<td>0.25</td>
<td>$6K$</td>
<td>$6K$</td>
<td>$6K$</td>
</tr>
<tr>
<td>0.10</td>
<td>$7K$</td>
<td>$7K$</td>
<td>$7K$</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Mean and Variance (Dependence)

\[ \text{PW}_{\text{Dep}}[X(T)] = X_0 + X \left[ \frac{1}{1 + i} + \frac{1}{(1 + i)^2} + \ldots + \frac{1}{(1 + i)^T} \right] \]

- \[ \text{PW}_{\text{Dep}}(A) = Y_{\text{Dep}} = -10K + X \left[ \frac{1}{1.08} + \frac{1}{(1.08)^2} + \frac{1}{(1.08)^3} \right] = -10K + X(P/A, 8, 3) \]
- From slide 15 we get
- \[ E[Y_{\text{Dep}}] = -10K + E[X] \cdot (P/A, 8, 3) \approx 2,885, \text{ as before} \]
- \[ \sigma_{Y,\text{Dep}}^2 = \sigma_X^2 [(P/A, 8, 3)]^2 = (1,140)^2 \cdot [2.5771]^2 \]
- or \[ \sigma_{Y,\text{Dep}} = (1,140) \cdot (2.5771) \approx 2,938 \]
Comparison

\[ \text{PW}_{\text{Ind}}[X(T)] = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + \ldots + \frac{X_T}{(1+i)^T} \]

\[ \text{PW}_{\text{Dep}}[X(T)] = X_0 + X\left[ \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^T} \right] \]

- In both the independent and dependent cases the mean values are the same ($2,885$).
- The standard deviation in the dependent case ($2,938$) is 73% larger than that of the independent case ($1,700$).
- \( P_{\text{Dep}}(\text{PW} < 0) = P(Z < - (2885/2938)) = P(Z < -0.98) = 0.16 \)
- Compare with \( P_{\text{Ind}}(\text{PW} < 0) = 0.04 \) (slide 32)
- These two cases are considered as bounding the problem.