For plane strain and small-scale yielding conditions:

\[ J = \frac{k^2}{E} \left(1 - \nu^2\right) \]

\[ \sigma_p = \frac{1}{3\pi} \left(\frac{k}{\sigma_y}\right)^2 \]

where \( \sigma_p \) is the monotonic plastic zone size.

J-dominance spans a distance \( R \) of up to \( \frac{1}{4} \sigma_p \):

\[ R = \frac{1}{12\pi} \left(\frac{k}{\sigma_y}\right)^2 \]

CTOD: \( \delta_e \approx 0.6 \frac{J}{\sigma_y} \) where \( J = \frac{k^2}{E} \left(1 - \nu^2\right) \)

\[ \delta_e = 0.6 \frac{k^2}{E\sigma_y} \]

We were given \( E = 500 \)

\[ \delta_e = 0.6 \frac{k^2}{500 \sigma_y^2} \]

\[ \delta_e = 0.6 \left(1 - 0.33^2\right) \frac{k^2}{500 \sigma_y^2} = \frac{0.6(1-0.33^2)}{500} \left(\frac{k}{\sigma_y}\right)^2 \]

Comparing \( R \) to \( \delta_e \):

\[ \frac{R}{\delta_e} = \frac{1}{12\pi} \left(\frac{k}{\sigma_y}\right)^2 = 24.8 \Rightarrow R = 25 \delta_e \]

J-dominance requirement satisfied.
$\sigma = 85 \text{ MPa}$

From Appendix A.1

\[ K_I = \sigma_{\text{yield}} f \left( \frac{\sigma}{\sigma_{\text{yield}}} \right) \rightarrow \frac{\sigma}{\sigma_{\text{yield}}} = 0.25 \]

\[ f \left( \frac{\sigma}{\sigma_{\text{yield}}} \right) = 2.665 \]

\[ K_I = 85 \text{ MPa} \sqrt{0.025 \text{ m}} \times 2.665 \]

\[ K_I = 35.82 \text{ MPa} \sqrt{\text{m}} \]

Assuming plane strain behavior:

\[ r_p = \frac{1}{3} \pi \left( \frac{K_I}{\sigma_{\text{yield}}} \right)^2 = 1.51 \text{ mm} \]

The region of J-dominance can be described as:

\[ R \approx \frac{1}{4} r_p = 3.775 \text{ mm} \]

Also, $r_o$ should be greater than the size of the process zone (e.g. page 311 of the text: the grain size for transgranular cleavage or intergranular fracture, and the mean spacing of void-nucleating particles for ductile failure by void growth).

Therefore, the region of validity for the material with a grain size of 22.0 mm is extremely small ($r_o \approx R$). Also, very few grains will fit into the above described region of validity ($R$). Therefore, crystal plasticity (rather than continuum) may dominate.
The region of validity of material 1, with a grain size of 10μm, is much larger and continuum assumptions will hold.

(2) For elastic-plastic fracture toughness testing, the depth of the initial crack must be at least one-half the width of the specimen to ensure a large region of J-dominance. As discussed in class, pure tension with no eccentricity of loading leads to a very small region of J-dominance. With increasing amounts of eccentricity (and therefore bending), the region of J-dominance increases in size. Computational results to support this were given in the notes.

\[ (P)(e) \] gives a measure of the bending moment.
Rotate the axes about the '3' direction by an angle \( \alpha = 90 - \beta \)

\[ \begin{pmatrix} 3' \end{pmatrix} \]

Transforming the stress tensor, using the direction cosine matrix \( \mathbf{D} \):

\[ \sigma_{ij}' = \mathbf{D}_{ij} \sigma_{ij} \]

where \( \mathbf{D}_{ij} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

\[ \sigma_{ij}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \sigma_{22}' = l_{22} l_{33} \sigma_{22} = (\cos^2 \alpha) \sigma \]

\( = (\sin^2 \beta) \sigma \)

\[ \sigma_{12}' = l_{21} l_{22} \sigma_{22} = (\sin \alpha)(\cos \alpha) \sigma \]

\( = (\cos \beta \sin \beta) \sigma \)

Globally, \( k_z = Y \sqrt{\frac{\sigma}{\sigma_{22}}} \), where \( Y \) is a geometric factor

Locally, \( k_z = Y (\sin^2 \beta) \sqrt{\frac{\sigma}{\sigma_{22}}} \)

\[ k_z = Y (\sin \beta)(\cos \beta) \sqrt{\frac{\sigma}{\sigma_{22}}} \]

\[ k_z = \sin \beta / k_z \]

\[ k_z = \sin \beta / k_z \]

\[ k_z = \sin \beta \cos \beta \]
11.5 Fracture surface asperity height = 0.75 mm

We can assume linear elastic conditions for the ceramic:

\[
J = \frac{k^2}{E} (1 - \nu^2) \rightarrow \text{assumed } \nu \approx 0.35
\]

We also know that \( \sigma_t = \frac{d_0 J}{G_0} \)

\( d_0 \approx 0.3 \) since \( n \approx 1 \)

Assume \( G_0 \) = the tensile rupture strength.

For a superimposed Mode I load, the maximum CTOD occurs at fracture, where \( J \rightarrow J_{\text{IC}} \):

\[
J_{\text{IC}} = \frac{k_{\text{IC}}^2}{E} (1 - \nu^2)
\]

At fracture, \( \sigma_t = 0.3 \left( \frac{3\text{MPa}}{3.75 \times 10^8 \text{MPa}} \right) \left( \frac{1 - 0.25^2}{2.75 \times 10^8 \text{MPa} \times 250 \text{MPa}} \right) \)

\( \sigma_t = 2.7 \times 10^{-5} \text{M} \)

\( \sigma_t = 2.7 \times 10^{-5} \text{mm} @ \text{fracture} \)

This much lower than the fracture surface asperity height.

The apparent fracture resistance in mode III will always be higher than that in mode I because the crack faces are not able to separate, which leads to closure locking. In pure mode I loading, the surfaces of the crack will separate and fracture will occur at a lower applied load. This reinforces the fact that mode I loading is most damaging.
Problem 5

$s = 0.2 \text{mm}$

$\sigma_y = 450 \text{MPa}$

$J_{IC} = 85 \times 10^3 \text{J/m}^2$

$E = 70,000 \text{Pa}$

$\nu = 0.3$

Valid $K_{IC}$ test:

Plane Strain: $f_p = \frac{1}{3\pi} \left( \frac{K_{IC}}{\sigma_y} \right)^2$

Recall: $J_{IC} = \frac{k_{IC}^2}{E} (1 - \nu^2) \Rightarrow K_{IC} = \sqrt{\frac{J_{IC} E}{1 - \nu^2}} = 80.8 \text{MPa} \sqrt{\text{m}}$

$f_p = \frac{1}{3\pi} \left( \frac{80.8 \text{MPa} \sqrt{\text{m}}}{450 \text{MPa}} \right)^2 = 3.43 \text{mm}$

$a, (w-a), b > 25 f_p = 86 \text{mm}$

Valid $J_{IC}$ test:

$a, (w-a), b > 25 \left( \frac{J_{IC}}{\sigma_y} \right) = 4.7 \text{mm}$

(3 much less severe requirements)