18.01 Solutions to Exam 1

Problem 1 (15 points) Use the definition of the derivative as a limit of difference quotients to compute the derivative of $y = x + \frac{1}{x}$ for all points $x > 0$. Show all work.

Solution to Problem 1 Denote by $f(x)$ the function $x + \frac{1}{x}$. By definition, the derivative of $f(x)$ at $x = a$ is,

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$ 

The increment $f(a + h) - f(a)$ equals,

$$\left( (a + h) + \frac{1}{a + h} \right) - \left( a + \frac{1}{a} \right) = h + \left( \frac{1}{a + h} - \frac{1}{a} \right).$$

To compute the second term, clear denominators,

$$\frac{1}{a+h} - \frac{1}{a} = \frac{1}{a+h} - \frac{1}{a} = \frac{1}{a} - \frac{1}{a} = \frac{1}{a} - \frac{1}{a} = \frac{a - (a + h)}{a(a + h)} = \frac{-h}{a(a + h)}.$$ 

Thus the increment $f(a + h) - f(a)$ equals,

$$h - \frac{h}{a(a + h)}.$$ 

Factoring $h$ from each term, the difference quotient equals,

$$\frac{f(a + h) - f(a)}{h} = 1 - \frac{1}{a(a + h)}.$$ 

Thus the derivative of $f(x)$ at $x = a$ equals,

$$f'(a) = \lim_{h \to 0} \left( 1 - \frac{1}{a(a + h)} \right) = 1 - \frac{1}{a(a + 0)} = 1 - \frac{1}{a}.$$ 

Therefore the derivative function of $f(x)$ equals,

$$f'(x) = 1 - \frac{1}{x^2}.$$
**Problem 2** (10 points) For the function $f(x) = e^{-x^2/2}$, compute the first, second and third derivatives of $f(x)$.

**Solution to Problem 2** Set $u$ equals $-x^2/2$ and set $v$ equals $e^u$. So $v$ equals $f(x)$. By the chain rule,

$$\frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx}.$$ 

Since $v$ equals $e^u$, $dv/du$ equals $(e^u)' = e^u$. Since $u$ equals $-x^2/2$, $du/dx$ equals $-(2x)/2 = -x$. Thus, back-substituting,

$$f'(x) = \frac{dv}{dx} = (e^u)(-x) = e^{-x^2/2}(-x) = -xe^{-x^2/2}.$$

For the second derivative, let $u$ and $v$ be as defined above, and set $w$ equals $-xv$. So $w$ equals $f'(x)$. By the product rule,

$$\frac{dw}{dx} = (-x)'v + (-x)v' = -v - x \frac{dv}{dx}.$$ 

By the last paragraph,

$$\frac{dv}{dx} = -xe^{-x^2/2}.$$ 

Substituting in,

$$f''(x) = \frac{dw}{dx} = -e^{-x^2/2} - x(-xe^{-x^2/2}) = -e^{-x^2/2} + x^2 e^{-x^2/2} = (x^2 - 1)e^{-x^2/2}.$$ 

For the third derivative, take $u$ and $v$ as above, and set $z$ equals $(x^2 - 1)v$. So $z$ equals $f''(x)$. By the product rule,

$$\frac{dz}{dx} = (x^2 - 1)'v + (x^2 - 1)v' = 2xv + (x^2 - 1) \frac{dv}{dx}.$$ 

By the first paragraph,

$$\frac{dv}{dx} = -xe^{-x^2/2}.$$ 

Substituting in,

$$f'''(x) = \frac{dz}{dx} = 2xe^{-x^2/2} + (x^2 - 1)(-xe^{-x^2/2}) = 2xe^{-x^2/2} + (-x^3 + x)e^{-x^2/2} = (-x^3 + 3x)e^{-x^2/2}.$$
Extra credit (5 points) Only attempt this after you have completed the rest of the exam and checked your answers. For every positive integer \( n \), show that the \( n^{th} \) derivative of \( f(x) \) is of the form \( f^{(n)}(x) = p_n(x)f(x) \), where \( p_n(x) \) is a polynomial. Also, give a rule to compute \( p_{n+1}(x) \), given \( p_n(x) \).

Solution to extra credit problem The claim, proved by induction on \( n \), is that for every positive integer \( n \), \( f^{(n)}(x) \) equals \( p_n(x) \) where \( p_n(x) \) is a degree \( n \) polynomial and,

\[
p_{n+1}(x) = -xp_n(x) + p'_n(x).
\]

The solution to Problem 2 proves this when \( n = 1, 2 \) and 3. Let \( n \) be a positive integer. By way of induction, assume the result is proved for \( n \). Precisely, assume \( f^{(n)}(x) \) equals \( p_n(x)e^{-x^2/2} \) where \( p_n(x) \) is a degree \( n \) polynomial. The goal is to prove the result for \( f^{(n+1)}(x) \); precisely, \( f^{(n+1)}(x) \) equals \( p_{n+1}(x)e^{-x^2/2} \) for a degree \( n + 1 \) polynomial \( p_{n+1}(x) \). By definition,

\[
f^{(n+1)}(x) = \frac{d}{dx}(f^{(n)}(x)).
\]

By the induction hypothesis, this equals,

\[
\frac{d}{dx}(p_n(x)e^{-x^2/2}).
\]

Let \( u \) and \( v \) be as above, and set \( y \) equals \( p_n(x)v \). So \( y \) equals \( f^{(n)}(x) \). By the product rule,

\[
\frac{dy}{dx} = p'_n(x)v + p_n(x)v' = p'_n(x)v + p_n \frac{dv}{dx}.
\]

As computed above,

\[
\frac{dv}{dx} = -xe^{-x^2/2}.
\]

Substituting in,

\[
\frac{dy}{dx} = p'_n(x)e^{-x^2/2} + p_n(x)(-xe^{-x^2/2}) = (-xp_n(x) + p'_n(x))e^{-x^2/2}.
\]

Since \( p_n(x) \) is a degree \( n \) polynomial, \( p'_n(x) \) is a degree \( n - 1 \) polynomial and \( -xp_n(x) \) is a degree \( n + 1 \) polynomial. Thus the sum \( -xp_n(x) + p'_n(x) \) is a degree \( n + 1 \) polynomial. Defining \( p_{n+1}(x) \) to be,

\[
p_{n+1}(x) = -xp_n(x) + p'_n(x),
\]

this gives,

\[
f^{(n+1)}(x) = \frac{dy}{dx} = p_{n+1}(x)e^{-x^2/2}.
\]

So the result for \( n + 1 \) follows from the result for \( n \). Therefore the result is proved by induction on \( n \). Moreover, this gives the inductive formula for \( p_n(x) \),

\[
p_{n+1}(x) = -xp_n(x) + p'_n(x).
\]
Problem 3 (15 points) A function \( y = f(x) \) satisfies the implicit equation,
\[
2x^3 - 9xy + 2y^3 = 0.
\]
The graph contains the point \((1, 2)\). Find the equation of the tangent line to the graph of \( y = f(x) \) at \((1, 2)\).

Solution to Problem 3 Differentiating both sides of the equation gives,
\[
\frac{d}{dx}(2x^3 - 9xy + 2y^3) = \frac{d}{dx}(0) = 0.
\]
Because the derivative is linear,
\[
\frac{d}{dx}(2x^3 - 9xy + 2y^3) = 2\frac{d(x^3)}{dx} - 9\frac{d(xy)}{dx} + 2\frac{d(y^3)}{dx}.
\]
Of course \(d(x^3)/dx\) equals \(3x^2\). By the product rule,
\[
\frac{d(xy)}{dx} = \frac{d(x)}{dx}y + x \frac{dy}{dx} = y + x \frac{dy}{dx}.
\]
For the last term, the chain rule gives,
\[
\frac{d(y^3)}{dx} = \frac{d(y^3)}{dy} \frac{dy}{dx} = 3y^2 \frac{dy}{dx}.
\]
Substituting in gives,
\[
\frac{d}{dx}(2x^3 - 9xy + 2y^3) = 2(3x^2) - 9\left(y + x \frac{dy}{dx}\right) + 2(3y^2) \frac{dy}{dx} = (6x^2 - 9y) + (6y^2 - 9x) \frac{dy}{dx}.
\]
By the first paragraph, \(d/dx(2x^3 - 9xy + 2y^3)\) equals 0. Substituting in gives the equation,
\[
(6x^2 - 9y) + (6y^2 - 9x) \frac{dy}{dx} = 0.
\]
Subtracting the first term from each side gives,
\[
(6y^2 - 9x) \frac{dy}{dx} = (9y - 6x^2).
\]
Dividing both sides by \((6y^2 - 9x)\) gives,
\[
\frac{dy}{dx} = \frac{9y - 6x^2}{6y^2 - 9x} = \frac{3y - 2x^2}{2y^2 - 3x}.
\]
Finally, plugging in $x$ equals 1 and $y$ equals 2 gives,

$$\frac{dy}{dx} = \frac{3(2) - 2(1)^2}{2(2)^2 - 3(1)} = \frac{6 - 2}{8 - 3} = \frac{4}{5}.$$ 

Therefore, the equation of the tangent line is,

$$y = \frac{4}{5}(x - 1) + 2,$$

which simplifies to,

$$y = \frac{4}{5}x + \frac{6}{5}.$$ 

**Problem 4** (20 points) The point (0, 4) is **not** on the graph of $y = x + 1/x$, but it is contained in exactly one tangent line to the graph.

(a) (15 points) Find the one value of $a$ for which the tangent line to the graph of $y = x + 1/x$ at $(a, a + 1/a)$ contains (0, 4).

**Hint:** You do not need to solve a quadratic equation to find $a$.

**Solution to (a)** By the **Solution to Problem 1** the derivative of $x + 1/x$ equals,

$$y' = 1 - \frac{1}{x^2}.$$ 

Thus the slope of the tangent line to the graph at $x = a$ is,

$$1 - \frac{1}{a^2} = \frac{a^2 - 1}{a^2}.$$ 

Therefore, the equation of the tangent line equals,

$$y = \frac{a^2 - 1}{a^2}(x - a) + \left( a + \frac{1}{a} \right) = \frac{a^2 - 1}{a^2}x + \frac{1 - a^2}{a} + \left( \frac{a^2 + 1}{a} \right).$$ 

This simplifies to give the equation,

$$y = \frac{a^2 - 1}{a^2}x + \frac{2}{a}.$$ 

By hypothesis, (0, 4) is contained in the tangent line. Plugging in $x = 0$ and $y = 4$ gives,

$$4 = \frac{(a^2 - 1)}{a^2}0 + \frac{2}{a} = \frac{2}{a}.$$ 

Multiplying both sides by $a$ gives,

$$4a = 2.$$ 

Dividing both sides by 4 gives,

$$a = 2/4 = \frac{1}{2}.$$
(b) (5 points) Write the equation of the corresponding tangent line.

Solution to (b) From the computation above, the equation of the tangent line at \( x = a \) is,

\[
y = \frac{(a^2 - 1)}{a^2}x + \frac{2}{a}.
\]

Plugging in \( a = 1/2 \) gives,

\[
a^2 - 1 = \frac{1}{4} - 1 = -\frac{3}{4},
\]

\[
\frac{a^2 - 1}{a^2} = \left(-\frac{3}{4}\right)(4) = -3,
\]

and,

\[
\frac{2}{a} = 2(2) = 4.
\]

Therefore the equation of the tangent line equals,

\[
y = -3x + 4.
\]

Problem 5 (25 points) In an automobile crash-test, a car is accelerated from rest at 2 \( \text{m/s}^2 \) for 5 seconds and then decelerated at \(-4\text{m/s}^2\) until it strikes a barrier. The position function is,

\[
s(t) = \begin{cases} 
\frac{t^2}{2} & 0 \leq t < 5 \\
-2t^2 + At + B & t \geq 5
\end{cases}
\]

(a) (10 points) Assuming that both \( s(t) \) and \( s'(t) \) are continuous at \( t = 5 \), determine \( A \) and \( B \).

Solution to (a) Because \( s(t) \) is continuous at \( t = 5 \), the left-hand limit and the right-hand limit are equal. The left-hand limit is,

\[
\lim_{t \to 5^-} s(t) = \lim_{t \to 5^-} \frac{t^2}{2} = 25.
\]

The right-hand limit is,

\[
\lim_{t \to 5^+} s(t) = \lim_{t \to 5^+} (-2t^2 + At + B) = -2(5)^2 + A(5) + B = -50 + 5A + B.
\]

This gives the equation,

\[
25 = -50 + 5A + B,
\]

which simplifies to,

\[
5A + B = 75.
\]

The derivative \( s'(t) \) equals,

\[
s'(t) = \begin{cases} 
\frac{(t^2)'}{2} & 0 \leq t < 5 \\
(-2t^2 + At + B)' & t \geq 5
\end{cases}
\]
which equals,
\[ s'(t) = \begin{cases} 
2t & 0 \leq t < 5 \\
-4t + A & t > 5
\end{cases} \]

Because \( s'(t) \) is continuous at \( t = 5 \), the left-hand limit and the right-hand limit are equal. The left-hand limit is,
\[ \lim_{t \to 5^-} s'(t) = \lim_{t \to 5^-} 2t = 2(5) = 10. \]
The right-hand limit is,
\[ \lim_{t \to 5^+} s'(t) = \lim_{t \to 5^+} (-4t + A) = -4(5) + A = -20 + A. \]
This gives the equation,
\[ 10 = -20 + A, \]
which simplifies to,
\[ A = 30. \]
Plugging in \( A = 30 \) to the first equation gives,
\[ 5(30) + B = 75, \]
which simplifies to,
\[ B = 75 - 5(30) = 75 - 150 = -75. \]
Therefore, the solution is,
\[ A = 30, \quad B = -75. \]

(b)(15 points) The barrier is located at \( s = 33 \) meters. Determine the velocity of the car when it strikes the barrier. (The quadratic polynomial has whole number roots.)

**Solution to (b)** For \( t > 5 \), the equation for displacement is,
\[ s(t) = -2t^2 + 30t - 75. \]
The moment \( T \) when the car strikes the barrier is the solution of the equation \( s(T) = 33 \),
\[ -2T^2 + 30T - 75 = 33. \]
Subtracting 33 from each side gives the equation,
\[ -2T^2 + 30T - 108 = 0. \]
Dividing each side by \(-2\) gives the equation,
\[ T^2 - 15T + 54 = 0. \]
The fraction \( \frac{54}{54} \) factors as \( 2 \times 27, \ 3 \times 18 \) and \( 6 \times 9 \). In the last case, the sum of the factors is +15. Thus the quadratic polynomial factors as,

\[
T^2 - 15T + 54 = (T - 6)(T - 9).
\]

The two possible solutions of \((T - 6)(T - 9) = 0\) are \( T = 6 \) and \( T = 9 \). Since the car cannot crash twice, the car crashes at the moment,

\[ T = 6. \]

For \( t > 5 \), the equation of \( v(t) = s'(t) \) was calculated above to be,

\[ s'(t) = -4t + A = -4t + 30. \]

Plugging in \( t = T = 6 \) gives,

\[ s'(6) = -4(6) + 30. \]

Therefore, at the moment the car crashes into the barrier, the velocity is,

6 meters/second.

**Problem 6** (15 points) For each of the following functions, compute the derivative. Show all work.

(a) (4 points) \( y = (e^x - e^{-x})/(e^x + e^{-x}) \)

**Solution to (a)** Set \( u = e^x - e^{-x} \) and \( v = e^x + e^{-x} \). Then \( y = u/v \). By the quotient rule, the derivative is,

\[
\frac{dy}{dx} = \frac{1}{v^2} \left( \frac{du}{dx} v - u \frac{dv}{dx} \right).
\]

Using the chain rule,

\[
\frac{du}{dx} = e^x(1) - e^{-x}(-1) = e^x + e^{-x} = v.
\]

Similarly,

\[
\frac{dv}{dx} = e^x(1) + e^{-x}(-1) = e^x - e^{-x} = u.
\]

Plugging in gives,

\[
\frac{dy}{dx} = \frac{1}{v^2} (v^2 - u^2).
\]

Expanding gives,

\[
v^2 - u^2 = (e^x - e^{-x})^2 - (e^x + e^{-x})^2 = [(e^x)^2 - 2e^xe^{-x} + (e^{-x})^2] - [(e^x)^2 + 2e^xe^{-x} + (e^{-x})^2].
\]

Cancelling, this gives,

\[ v^2 - u^2 = -4e^xe^{-x} = -4. \]
Therefore, the derivative equals,
\[ \frac{dy}{dx} = \frac{-4}{v^2} = \frac{-4}{(e^x - e^{-x})^2}. \]

(b) (3 points) \( y = x \ln(x) - x \)

**Solution to (b)** Because the derivative is linear,
\[ y' = (x \ln(x))' - (x)' = (x \ln(x))' - 1. \]

By the product rule,
\[ (x \ln(x))' = (x)' \ln(x) + x(\ln(x))' = 1 \ln(x) + x \frac{1}{x} = \ln(x) + 1. \]

Therefore the derivative is \( \ln(x) + 1 - 1 \), which is,
\[ y' = \ln(x). \]

(c) (3 points) \( y = \sqrt{1 + x^{1234}} \)

**Solution to (c)** Set \( u = x^{1234} \). Set \( v = 1 + u \), which equals \( 1 + x^{1234} \). Then \( y = v^{1/2} \), which equals \( (1 + x^{1234})^{1/2} \). By the chain rule,
\[ \frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{du} \frac{du}{dx}. \]

By the formula for the derivative of \( x^a \),
\[ \frac{du}{dx} = 1234x^{1233}, \quad \frac{dv}{du} = 1, \quad \frac{dy}{dv} = \frac{1}{2} v^{-1/2}. \]

Thus the chain rule gives,
\[ \frac{dy}{dx} = \frac{1}{2} v^{-1/2}(1)(1234x^{1233}) = \frac{1}{2}(1 + x^{1234})^{-1/2}(1234x^{1233}). \]

This simplifies to give,
\[ y' = 617x^{1233}/\sqrt{1 + x^{1234}}. \]

(d) (5 points) \( y = \log_{10}(x^3 + 3x) \).

**Solution to (d)** The inner term factors as \( x^3 + 3x = x(x^2 + 3) \). Since \( \log_{10}(AB) = \log_{10}(A) + \log_{10}(B) \), the expression for \( y \) simplifies to,
\[ y = \log_{10}(x(x^2 + 3)) = \log_{10}(x) + \log_{10}(x^2 + 3). \]
Because the derivative is linear,

\[ y' = (\log_{10}(x))' + (\log_{10}(x^2 + 3))'. \]

The formula for the derivative of a logarithm function is,

\[ \frac{d(\log_a(x))}{dx} = \frac{1}{\ln(a)x}. \]

Thus,

\[ (\log_{10}(x))' = \frac{1}{\ln(10)x}. \]

For the second term, set \( u \) equals \( x^2 + 3 \). And set \( v \) equals \( \log_{10}(u) = \log_{10}(x^2 + 3) \). By the chain rule,

\[ \frac{d}{dx}(\log_{10}(x^2 + 3)) = \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}. \]

By the formula for the derivative of a logarithm function,

\[ \frac{dv}{du} = \frac{d}{du}(\log_{10}(u)) = \frac{1}{\ln(10)u}. \]

And, of course,

\[ \frac{du}{dx} = (x^2 + 3)' = 2x. \]

Thus, the derivative is,

\[ \frac{dv}{dx} = \frac{1}{\ln(10)u} \cdot (2x) = \frac{1}{\ln(10)(x^2 + 3)} \cdot (2x). \]

Putting the pieces together,

\[ y' = \frac{1}{\ln(10)x} + \frac{2x}{\ln(10)(x^2 + 3)}. \]

This simplifies to give,

\[ y' = \frac{3(x^2 + 1)}{(\ln(10)x(x^2 + 3))}. \]