18.01 Exam 5

Problem 1 (25 points) Use integration by parts to compute the antiderivative,
\[ \int x^3 e^{-x^2} dx. \]

Solution to Problem 1 The derivative of \( e^{-x^2} \) is \(-2xe^{-x^2}\). Thus, set
\[
\begin{align*}
u &= x^2, & dv &= xe^{-x^2} dx, \\
du &= 2xdx, & v &= -e^{-x^2}/2.
\end{align*}
\]
Then, by integration by parts,
\[
\int u dv = uv - \int v du,
\]
\[
\int x^3 e^{-x^2} dx = -\frac{1}{2} x^2 e^{-x^2} + \int xe^{-x^2} dx.
\]
As computed above, the derivative of \( e^{-x^2} \) is \(-2xe^{-x^2}\). Thus the new integral is \(-e^{-x^2}/2 + C\). Therefore,
\[
\int x^3 e^{-x^2} dx = -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} = -(x^2 + 1)e^{-x^2}/2.
\]

Problem 2 (20 points) Use polynomial division and factoring to compute the antiderivative,
\[ \int \frac{x^3 - 1}{x^2 - 3x + 2} dx. \]
You will not need to use partial fractions (though you are free to do so).

Solution to Problem 2 By the polynomial division algorithm,
\[ x^3 - 1 = (x + 3)(x^2 - 3x + 2) + (7x - 7). \]
Thus the fraction is,
\[ \frac{x^3 - 1}{x^2 - 3x + 2} = x + 3 + \frac{7(x - 1)}{x^2 - 3x + 2}. \]
The denominator of the new fraction factors as,

\[ x^2 - 3x + 2 = (x - 1)(x - 2). \]

Thus the original fraction is,

\[ x + 3 + \frac{7(x - 1)}{(x - 1)(x - 2)} = x + 3 + \frac{7}{x - 2}. \]

Therefore the antiderivative is,

\[ \int x + 3 + \frac{7}{x - 2} \, dx = \frac{1}{2}x^2 + 3x + 7 \ln(|x - 2|) + C. \]

**Problem 3** (20 points)

(a) (15 points) Find the partial fraction decomposition of,

\[ \frac{x + 3}{x^2 - 2x + 1}. \]

**Solution to (a)** Using the quadratic formula, \( x^2 - 2x + 1 \) has only the root 1. Thus it is,

\[ x^2 - 2x + 1 = (x - 1)^2. \]

So the partial fraction decomposition has the form,

\[ \frac{x + 3}{x^2 - 2x + 1} = \frac{A}{(x - 1)^2} + \frac{B}{x - 1}, \]

for some choice of \( A \) and \( B \). By the Heaviside cover-up method, \( A \) equals,

\[ (x + 3)_{x=1} = 4. \]

Thus the partial fraction decomposition is,

\[ \frac{x + 3}{x^2 - 2x + 1} = \frac{4}{(x - 1)^2} + \frac{B}{x - 1}. \]

Plugging in \( x = 0 \) gives,

\[ 3 = \frac{x + 3}{x^2 - 2x + 1} \bigg|_{x=0} = \frac{4}{(0 - 1)^2} + \frac{B}{0 - 1} = 4 - B. \]

Therefore \( B \) equals 1. So the partial fraction decomposition is,

\[ \frac{x + 3}{x^2 - 2x + 1} = \frac{4}{(x - 1)^2} + \frac{1}{x - 1}. \]
(b)(5 points) Use your answer from (a) to compute the antiderivative of

$$\int \frac{x + 3}{x^2 - 2x + 1} \, dx.$$  

**Solution to (b)** By the Solution to (a) the integral is,

$$\int \frac{x + 3}{x^2 - 2x + 1} \, dx = \int \frac{4}{(x - 1)^2} + \frac{1}{x - 1} \, dx.$$  

The second integral is easily computed and gives,

$$\int \frac{x + 3}{x^2 - 2x + 1} \, dx = -4/(x - 1) + \ln(|x - 1|) + C.$$  

**Problem 4 (25 points)** Let $a$ be a positive real number.

(a)(3 points) Determine the range of $x$ on which $2ax - x^2$ is nonnegative.

**Solution to (a)** Completing the square gives,

$$-x^2 + 2ax = -(x - a)^2 + a^2.$$  

Therefore the expression is nonnegative when,

$$-(x - a)^2 + a^2 \geq 0 \iff a^2 \geq (x - a)^2.$$  

Taking square roots, the expression is nonnegative when,

$$a \geq x - a \geq -a.$$  

This simplifies to,

$$0 \leq x \leq 2a.$$  

(b)(22 points) For that range, compute the antiderivative,

$$\int \sqrt{2ax - x^2} \, dx.$$  

**Solution to (b)** Begin by making the linear change of variables,

$$u = x - a, \quad du = dx.$$  

The new integral is,

$$\int \sqrt{a^2 - u^2} \, du.$$
This integral can be computed using an inverse trigonometric substitution,

\[ u = a \sin(\theta), \quad du = a \cos(\theta) d\theta. \]

Plugging in, the new integral is,

\[ \int \sqrt{a^2(1 - \sin^2(\theta))} \, (a \cos(\theta) d\theta) = a^2 \int \cos^2(\theta) d\theta. \]

This can be solved either using integration by parts or using the half-angle formula from trigonometry. The half-angle formula is decidedly faster and gives,

\[ \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}. \]

Substituting this in, the new integral is,

\[ a^2 \int \frac{1}{2} + \frac{1}{2} \cos(2\theta) d\theta = a^2 \left( \frac{\theta}{2} + \frac{1}{4} \sin(2\theta) \right) + C. \]

Using the double-angle formula, this simplifies to,

\[ a^2 \left( \frac{\theta}{2} + \frac{1}{2} \sin(\theta) \cos(\theta) \right) + C. \]

Back-substituting for \( u \) gives,

\[ \frac{a^2}{2} \sin^{-1}(u/a) + \frac{1}{2} u \sqrt{a^2 - u^2} + C. \]

Back-substituting for \( x \) gives,

\[ \int \sqrt{2ax - x^2} \, dx = \left( \frac{a^2}{2} \sin^{-1}(u/a) + (x - a) \sqrt{2ax - x^2} \right) / 2 + C. \]

**Problem 5** (10 points) Compute the following derivatives. Please show all work to receive full credit.

(a) (5 points)

\[ y = \sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \frac{dy}{dx} = ? \]

**Solution to (a)** The derivative should be with respect to \( t \). Using the chain rule, it is,

\[ \frac{d}{dt} \left( \frac{e^t - e^{-t}}{2} \right) = \left( \frac{1}{2} e^t - \frac{1}{2} (-e^{-t}) \right) = \cosh(t). \]
(b) (5 points)

\[ y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}), \quad \frac{dy}{dx} = ? \]

There are at least 2 methods. You may use the one you prefer.

**Solution to (b)** The formula for the derivative of an inverse function is,

\[ \frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))}. \]

As computed above, the derivative of \( \sinh(x) \) is \( \cosh(x) \). Thus the derivative is,

\[ \frac{1}{\cosh(\sinh^{-1}(x))}. \]

Using the identity \( \cosh^2(x) - \sinh^2(x) = 1 \), the denominator is,

\[ \cosh(\sinh^{-1}(x)) = \sqrt{1 + [\sinh(\sinh^{-1}(x))]^2} = \sqrt{1 + x^2}. \]

Therefore the derivative is,

\[ \frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{1 + x^2}}, \]

**Extra credit** (5 points)

\[ y = \sin(\tan^{-1}(2x)), \quad \frac{dy}{dx} = ? \]

Only solutions in simplest terms will be accepted.

**Solution to the extra credit problem** Denote by \( \theta \) the angle \( \tan^{-1}(2x) \). There is a right triangle with angle \( \theta \) having opposite side \( 2x \) and adjacent side \( 1 \). Therefore the hypotenuse has length,

\[ \sqrt{(2x)^2 + 1^2} = \sqrt{4x^2 + 1}. \]

Since \( \sin(\theta) \) is the ratio of the opposite side by the hypotenuse,

\[ \sin(\tan^{-1}(2x)) = \frac{2x}{\sqrt{4x^2 + 1}}. \]

Using the quotient rule and the chain rule,

\[ \frac{d}{dx} \left( \frac{2x}{\sqrt{4x^2 + 1}} \right) = \frac{1}{4x^2 + 1} \left( (2)\sqrt{4x^2 + 1} - (2x) \left( \frac{1}{2\sqrt{4x^2 + 1}} \right) (8x) \right) = \frac{2}{(4x^2 + 1)^{3/2}}(4x^2 + 1 - 4x^2). \]

Therefore the derivative is,

\[ \frac{d}{dx} \sin(\tan^{-1}(2x)) = \frac{2}{(4x^2 + 1)^{3/2}}. \]