18.02: Practice Exam 3A

1. Let \((\bar{x}, \bar{y})\) be the center of mass of the triangle, with vertices at \((-2, 0), (0, 1), (2, 0)\) and uniform density \(\delta = 1\).
   a) Write an integral formula for \(\bar{y}\). Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.
   b) Find \(\bar{x}\).

2. Find the polar moment of inertia of the unit disk with density equal to the distance from the \(y\)-axis.

3. Let \(\mathbf{F} = (ax^2 + y^3 + 1)\mathbf{i} + (2x^3 + bxy^2 + 2)\mathbf{j}\) be a vector field, where \(a\) and \(b\) are constants.
   a) Find the values of \(a\) and \(b\) for which \(\mathbf{F}\) is conservative.
   b) For these values of \(a\) and \(b\), find \(f(x, y)\) such that \(\mathbf{F} = \nabla f\).
   c) Still using the values of \(a\) and \(b\) from part (a), compute \(\int_C \mathbf{F} \cdot d\mathbf{r}\) along the curve \(C\) such that \(x = e^t \cos t, y = e^t \sin t, 0 \leq t \leq \pi\).

4. For \(\mathbf{F} = yx^3\mathbf{i} + y^2\mathbf{j}\) find \(\int_C \mathbf{F} \cdot d\mathbf{r}\) on the portion of the curve \(y = x^2\) from \((0, 0)\) to \((1, 1)\).

5. Consider the region \(R\) in the first quadrant bounded by the curves \(y = x^2, y = x^2/5, xy = 2, \) and \(xy = 4\).
   a) Compute \(dxdy\) in terms of \(du dv\) if \(u = x^2/y\) and \(v = xy\).
   b) Find a double integral for the area of \(R\) in \(uv\) coordinates and evaluate it.

6. a) Let \(C\) be a simple closed curve going counterclockwise around a region \(R\). Let \(M = M(x, y)\). Express \(\int_C M dx\) as a double integral over \(R\).
   b) Find \(M\) so that \(\int_C M dx\) is the mass of \(R\) with density \(\delta(x, y) = (x + y)^2\).

7. Consider the region \(R\) enclosed by the \(x\)-axis, \(x = 1\) and \(y = x^3\).
   Travelling in a counterclockwise direction along the boundary \(C\) or \(R\), call \(C_1\) the portion of \(C\) that goes from \((0, 0)\) to \((0, 1)\), \(C_2\) the portion that goes from \((1, 0)\) to \((1, 1)\) and \(C_3\) the portion that goes from \((1, 1)\) to \((0, 0)\).
   a) Find the total work of \(\mathbf{F} = (1 + y^2)i\) around the boundary \(C\) of \(R\), in a counterclockwise direction.
   b) Calculate the work of \(\mathbf{F}\) along \(C_1\) and \(C_2\).
   c) Use parts (a) and (b) to find the work along the third side \(C_3\).
18.02 Practice Exam 3A Solutions

1. a) Area of triangle is base times height = 2, so \( \bar{y} = \frac{1}{2} \int_0^1 \int_{2y-2}^{2-2y} y \, dx \, dy \)
   
   b) By symmetry \( \bar{x} = 0 \)

2. \( \delta = |x| = r \cos \theta \). \( I_0 = \int_D \int_D r^2 \delta \, rdr \, \theta = 4 \int_0^{\pi/2} \int_0^1 r^4 \cos \theta \, dr \, \theta = 4 \int_0^{\pi/2} \frac{1}{5} \cos \theta \, d\theta = \frac{4}{5} \)

3. a) \( N_x = 6x^2 + by^2, M_y = ax^2 + 3y^2 \). \( N_x = M_y \) provided \( a = 6 \) and \( b = 3 \).
   
   b) \( f_x = 6x^2y + y^3 + 1 \implies f = 2x^3y + xy^3 + x + c(y). \) Therefore, \( f_y = 2x^3 + 3xy^2 + c'(y) \). Setting this equal to \( N \), we have \( 2x^3 + 3xy^2 + c'(y) = 2x^3 + 3y^2 + 2 \) so \( c'(y) = 2 \) and \( c = 2y \) (+constant). In all,
   
   \[ f = 2x^3y + xy^3 + x + 2y \quad (+\text{constant}) \]

   c) \( C \) starts at \((1,0)\) and ends at \((-e^\pi,0)\), so \( \int_C \mathbf{F} \cdot d\mathbf{r} = f(-e^\pi,0) - f(1,0) = -e^\pi - 1 \).

4. \( \int_C xy^3 \, dx + y^2 \, dy = \int_0^1 x^2 x^3 \, dx + (x^2)^2(2x \, dx) = \int_0^1 3x^5 \, dx = 1/2 \)

5. a) \[ \begin{vmatrix} u_x & u_y \\
             v_x & v_y 
        \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\
                        y & x 
        \end{vmatrix} = 3x^2/y. \] Therefore,
   
   \[ dudv = (3x^2/y) \, dxdy = 3u \, dxdy \implies dxdy = \frac{1}{3u} \, dudv \]

   b) \( \int_2^4 \int_1^5 \frac{1}{3u} \, dudv = \int_2^4 \frac{1}{3} \ln 5 \, dv = \frac{2}{3} \ln 5 \)

6. a) \( \int_C M \, dx = \int_R -M_y \, dA \)

   b) We want \( M \) such that \(-M_y = (x+y)^2\). Use \( M = -\frac{1}{3}(x+y)^3 \)

7. a) For \( \mathbf{F} \), \( M_y = 2y \) and \( N_x = 0 \), hence \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_R -2y \, dA = \int_0^1 \int_0^1 -2y \, dy \, dx = \int_0^1 -x^6 \, dx = -\frac{1}{7}. \)

   b) For the work through \( C_1 \), we have \( \mathbf{F} \cdot \mathbf{i} = 1 + y^2 = 1 + 0 = 1 \). The length of \( C_1 \) is 1, so the total work through \( C_1 \) is 1.

   The work through \( C_2 \) is zero because \( \mathbf{F} \cdot \mathbf{j} = 0 \).

   c) \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{7} - 1 - 0 = -\frac{8}{7} \)