TWELFTH HOMEWORK, PRACTICE PROBLEMS

1. Let \( \vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the vector field given by

\[
\vec{F}(x, y, z) = ay^2 \hat{j} + (by^2 + z^2) \hat{k}.
\]

(i) For which values of \(a\) and \(b\) is the vector field \(\vec{F}\) conservative?

(ii) Find a function \(f : \mathbb{R}^3 \rightarrow \mathbb{R}\) such that \(\vec{F} = \text{grad } f\), for these values.

(iii) Find the equation of a surface \(S\) with the property that for every smooth oriented curve \(C\) lying on \(S\),

\[
\int_C \vec{F} \cdot d\vec{s} = 0,
\]

for these values.

2. Let \(S\) be the rectangle with vertices \((0, 0, 0), (1, 0, 0), (1, 2, 2)\) and \((0, 2, 2)\). Find the flux of the vector field \(\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3\), given by

\[
\vec{F}(x, y, z) = y^2 \hat{i} - z^2 \hat{j} + x^2 \hat{k},
\]

through \(S\) in the direction of the unit normal vector \(\hat{n}\), for which \(\hat{n} \cdot \hat{k} > 0\).

3. Let \(C_a(P)\) be the circle of radius \(a\) centered at \(P\) and oriented counter-clockwise. A smooth rotation free vector field \(\vec{F}\) is defined on the whole of \(\mathbb{R}^2\), except for the points \(P_0 = (0, 0), P_1 = (4, 0),\) and \(P_3 = (8, 0)\), and

\[
\int_{C_2(P_0)} \vec{F} \cdot d\vec{s} = -2, \quad \int_{C_6(P_1)} \vec{F} \cdot d\vec{s} = 1 \quad \text{and} \quad \int_{C_{10}(P_3)} \vec{F} \cdot d\vec{s} = 3.
\]

Find the following line integrals.

(a)

\[
\int_{C_1(P_1)} \vec{F} \cdot d\vec{s}.
\]

(b)

\[
\int_{C_2(P_2)} \vec{F} \cdot d\vec{s}.
\]

(c)

\[
\int_{C_6(P_2)} \vec{F} \cdot d\vec{s}.
\]

4. (6.3.16)

5. (7.1.4)
6. (7.1.20)
7. (7.2.13)
9. (7.2.17)
10. (7.3.11)
11. (7.3.13)
12. (7.3.16)
13. (7.3.18)
14. (7.3.19)

Just for fun: Let $\omega$ be a $k$-form on $\mathbb{R}^n$. Show that

$$d(d\omega) = 0.$$ 

This basic fact about $d$ is often expressed in the formula $d^2 = 0$.  