

18.03 Class 13, March 3, 2010

Forced linear equations

- [1] Superposition II
- [2] Harmonic sinusoidal response
- [3] Exponential response formula
- [4] Sinusoidal response using complex replacement

I drew the spring/mass/dashpot system and added a force to it:
the little sail comes back into play.

$$mx'' + bx' + kx = F_{\text{ext}} \quad (*)$$

Notice by the way that I can put the damper on the left in parallel with the spring: it still opposes velocity. If $x' < 0$, $F_{\text{dash}} > 0$ and so on: so you get exactly the same equation.

Also important will be the "associated homogeneous equation"

$$mx'' + bx' + kx = 0 \quad (*)_h$$

which we know all about after Lecture 12. Final comment on this:
We can "reduce" this by dividing by m :

$$x'' + (b/m)x' + (k/m) = 0$$

If $b = 0$ we get solutions with circular frequency $\sqrt{k/m}$.
If $b > 0$, you get exponentially damped sinusoids, with smaller circular frequency ω_d (or not oscillating at all, if b is big enough).
In general, even if $b > 0$, we call $\sqrt{k/m}$ the "natural circular frequency" of the system, and write ω_n for it. So in the underdamped case, when there is an ω_d , $\omega_d < \omega_n$. So the reduced homogeneous equation is $x'' + (b/m)x' + \omega_n^2 = 0$.

[1] The general strategy in finding solutions is based on "superposition."

[Slide:]

Superposition I: If x_1 and x_2 are solutions of a homogeneous linear equation, then so is any linear combination $c_1 x_1 + c_2 x_2$.
If the equation is of second order and neither of x_1 , x_2 is a multiple of the other, then $c_1 x_1 + c_2 x_2$ is the general solution.

Now we have:

Superposition II: If x_p is any solution to $(*)$ and x_h is a solution to $(*)_h$, then $x_p + x_h$ is again a solution to $(*)$.

Proof: Plug x into $(*)$:

$$\begin{array}{ll} k] & x = x_p + x_h \\ b] & x' = x_p' + x_h' \\ m] & x'' = x_p'' + x_h'' \end{array}$$

$$\begin{aligned}
 mx'' + bx' + kx &= (m xp'' + b xp' + k xp) + (m xh'' + b xh' + k xh) \\
 &= F_{\text{ext}} + 0
 \end{aligned}$$

as we wanted.

In fact, if x_h is the general solution to $(*)_h$ then $x_p + x_h$ is the general solution to $(*)$.

This proof works for linear equations of any order; for example first order, where we already saw it as a consequence of the method of integrating factors.

Superposition II splits the problem of finding the general solution to $(*)$ into two parts:

- (1) find SOME solution to $(*)$, a "particular solution," and then
- (2) find the general solution of $(*)_h$.

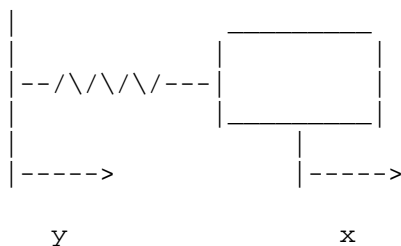
We have just been studying solutions too $(*)_h$, so let's focus on finding a particular solution.

[2] First case: $b = 0$, sinusoidal driving term.

The model is the weight on the rubber band. I showed its natural frequency, and then drove it by moving my hand sinusoidally, first slowly and then faster. At first the mass just follows the motion of my hand, but as the frequency picks up the rubber band starts to stretch at the lows and compress at the highs. At a certain point things get really wild, but then settle down - but now the mass is moving UP when my hand goes down, and vice versa.

Can we explain all that mathematically?

I drew a spring/mass system, forced through the spring:



Set up the parameters x and y so that when $x = y$ the spring is relaxed. So when $x > y$ the spring is stretched and the spring force is negative:

$$m x'' = k(y - x)$$

or $mx'' + kx = ky$

Think of y as the input signal - the position of my hand - and x as the system response or output signal.

So in the sinusoidal case, set the clock so that the input signal has the

$$y = A \cos(\omega t)$$

So : $x'' + \omega_n^2 x = A \omega_n^2 \cos(\omega t)$ (**)

There are two frequencies here: the natural frequency ω_n of the system and the frequency ω of the input signal $A \cos(\omega t)$.

The experiment with the weight and rubber and encourages us to use the Method of Optimism, and hope for a solution of the form

$$x_p = B \cos(\omega t)$$

To see what B must be, plug this into (**):

$$\begin{array}{r} \omega_n^2] \\ 1] \end{array} \quad \begin{array}{l} x_p = B \cos(\omega t) \\ x_p'' = -B \omega^2 \cos(\omega t) \end{array}$$

$$A \omega_n^2 \cos(\omega t) = B(\omega_n^2 - \omega^2) \cos(\omega t)$$

This works out if we take

$$B = \left(\frac{\omega_n^2}{\omega_n^2 - \omega^2} \right) A$$

The output amplitude is a multiple of the input amplitude, and the ratio is the GAIN:

$$H(\omega) = |B/A| = \frac{\omega_n^2}{|\omega_n^2 - \omega^2|}$$

Imagine the natural frequency of the oscillator fixed, and we slowly increase the frequency of the input signal. The graph of the gain starts when $\omega = 0$ at $H = 1$ - consistent with what we saw - and then increases to a vertical asymptote at $\omega = \omega_n$. This is RESONANCE, and then no such sinusoidal solution exists. There are solutions, of course, and we will come back to this case later. What happens with the weight and rubber band is that the nonlinear character of the spring asserts itself for large amplitude.

When $\omega > \omega_n$, the gain falls back towards zero.

Also: when $\omega < \omega_n$ the denominator is positive, and the output is a positive multiple of the input. When $\omega > \omega_n$ the denominator is negative, and the output signal is a negative multiple of the input: this is PHASE REVERSAL.

We have nicely accounted for all the things we observed. I demonstrated this effect using the Mathlet Harmonic Frequency Response: Variable Input Frequency.

[3] Exponential signals

$$mx'' + bx' + kx = A e^{rt} \quad (*)$$

We want to find some solution, x_p . Again as in the first order case,

try for a solution of the form $x_p = B e^{rt}$:

$$k] \quad x_p = B e^{rt}$$

$$b] \quad x_p' = B r e^{rt}$$

$$m] \quad x_p'' = B r^2 e^{rt}$$

$$A e^{rt} = B (m r^2 + b r + k) e^{rt} = B p(r) e^{rt}$$

We saw the same thing in the first order case. It's a general fact about linear constant coefficient equations with exponential right hand side:

$$a_n x^{(n)} + \dots + a_1 x' + a_0 x = A e^{rt} \quad (*)$$

with characteristic polynomial $p(s) = a_n s^n + \dots + a_1 s + a_0$:

Exponential Response Formula: A (particular) solution to (*) is given by:

$$x_p = A e^{rt} / p(r)$$

as long as $p(r)$ is not zero.

- i.e., if r is *not* a root of the characteristic polynomial.
If $p(r) = 0$, the system is "in resonance" with the input signal,
and there is no solution of the form $B e^{rt}$.

$$\text{Eg} \quad x'' + x' + 2x = 4 e^{3t} \quad p(3) = 3^2 + 3 + 2 = 14 \quad \text{so}$$

$$x_p = (4 / 14) e^{3t} .$$

$$[4] \text{ Now let's try this: } \quad x'' + x' + 2x = \cos(t)$$

by *complex replacement*: This is the real part of

$$z'' + z' + 2z = e^{it}$$

$$p(s) = s^2 + s + 2, \quad p(i) = (i)^2 + i + 2 = 1 + i,$$

$$z_p = (1/(1+i)) e^{it}$$

We want the real part.

$$z_p = ((1-i)/2) (\cos(t) + i \sin(t))$$

$$x_p = (1/2) (\cos(t) + \sin(t))$$

To understand this, sketch the triangle: sides $1/2$ and $1/2$ so
hypotenuse $A = \sqrt{2}/2$ and angle $\phi = \pi/4$:

$$x_p = (\sqrt{2}/2) \cos(t - \pi/4) .$$

Lesson: the damping term, b , creates a nonzero imaginary part in the denominator of the ERF, and that causes phase lag in the solution. It also causes the magnitude of the denominator to increase, and hence the amplitude of the sinusoidal solution to decrease.

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