

18.03 Class 20, March 19, 2010

Periodic signals, Fourier series

- [1] Periodic functions
- [2] Sines and cosines
- [3] Parity
- [4] Integrals

[1] Periodic functions: for example the heartbeat, or the sound of a violin, or innumerable electronic signals. I showed an example of somewhat simplified waveforms of a violin and a flute.

A function  $f(t)$  is "periodic" if there is  $L > 0$  such that  $f(t+2L) = f(t)$  for every  $t$ .  $2L$  is a "period."

So strictly speaking the examples given are not periodic, but rather they coincide with periodic functions for some period of time. Our methods will accept this approximation, and yield results which merely approximate real life behavior, as usual.

The constant function is periodic of every period. Otherwise, all the periodic functions we'll encounter have a minimal period, which is often called THE period.

Any "window" (interval) of length  $2L$  determines the function. You can choose the window as convenient. We'll often use the window  $[-L, L]$ . (This is one reason why chose to write the period as  $2L$ .)

[2] Sine and cosines are basic periodic functions. For this reason a natural period to start with is  $2L = 2\pi$  :  $L = \pi$ .

We'll use the basic window  $[-\pi, \pi]$ .

Question: what other cosines have period  $2\pi$  ?

1.  $\cos(t)$ ,  $\cos(t/2)$ ,  $\cos(t/3)$ , ...
2.  $\cos(\pi t)$ ,  $\cos(2\pi t)$ , ...
3.  $\cos(t)$ ,  $\cos(2t)$ ,  $\cos(3t)$ , ...

Answer:  $\cos(nt)$  for  $n = 1, 2, 3, \dots$  has minimal period  $2\pi/n$ .

I sketched graphs of  $\cos(t)$ ,  $\cos(2t)$ , and  $\cos(3t)$ .

Among the sines, we have  $\sin(t)$ ,  $\sin(2t)$ , ...

These are "harmonics" of the "fundamental" sinusoids with  $n = 1$ .

If  $f(t)$  and  $g(t)$  are periodic of period  $2L$  then so is  $af(t) + bg(t)$ .

So we can form linear combinations:

$$f(t) = \underbrace{a_0/2 + a_1 \cos(t) + a_2 \cos(2t) + \dots}_{\text{cosine series}} + \underbrace{b_1 \sin(t) + b_2 \sin(2t) + \dots}_{\text{sine series}} \quad (*)$$

This is a "Fourier Series." The  $a_n$  and  $b_n$  are the "Fourier Coefficients."

We'll see why the odd choice of  $a_0/2$  for the constant term shortly.

Theorem. Any reasonable [piecewise continuous] function of period  $2\pi$  has exactly one expression as a Fourier series.

The \*definition\* of the Fourier coefficients of a function  $f(t)$  is this: they are the coefficients that make (\*) true.

I set up the Mathlet FourierCoefficients and showed multiples of  $\cos(2t)$ , and then added multiples of  $\cos(t)$  - you can see the addition of functions.

For example, the standard squarewave is periodic of period  $2\pi$  and

$$\begin{aligned} Sq(t) &= 1 \quad \text{for } 0 < t < \pi \\ &= -1 \quad \text{for } -\pi < t < 0 \\ &= 0 \quad \text{for } t = 0, \pi \end{aligned}$$

The applet actually shows  $(\pi/4) Sq(t)$ . I tried to approximate it using cosines; but they are even and  $Sq$  is odd, so it doesn't look so good. Working with sines, it seems that the even ones are not useful - they break the symmetry round  $\pi/2$  exhibited by  $Sq(t)$ . I got pretty good results with  $b_1 = 1.00$ ,  $b_3 = .33$ ,  $b_5 = .20$ .

[3] Parity. A function  $f(t)$  is "even" if  $f(-t) = f(t)$ , odd if  $f(-t) = -f(t)$ .

Even + even is even, Odd + odd is odd; Even + Odd can be anything.

Even x even is even. 5 times 7 is odd, so I ask:

Q2. The product of two odd functions is

1. Even
2. Odd
3. Could be either or neither

Well, if  $f(t)$  and  $g(t)$  are odd then

$$f(-t)g(-t) = (-f(t))(-g(t)) = f(t)g(t)$$

so the product is even, in contrast to products of odd numbers. Also even times odd is odd.

Linear combinations of evens are even, of odds are odd.

$\cos(nt)$  is even,  $\sin(nt)$  is odd.

The only function which is both even and odd is the zero function. For  $f(t) = f(-t)$  and  $f(t) = -f(-t)$  together imply that  $f(t) = -f(t)$ .

If a periodic function  $f(t)$  is even, then  $f(t)$  - (cosine series) is a linear combination of evens and hence even, but it's also (sine series) and so odd, so it's zero, so:

The Fourier series of an even function is a cosine series:  $b_n = 0$ .

The Fourier series of an odd function is a sine series:  $a_n = 0$

[The same argument shows that if a polynomial is even then it's a sum of even powers of  $t$ ; if it's odd then it's a sum of odd powers of  $t$ .]

[4] Average. The average of a function of period  $2\pi$  is

$$\text{Ave}(f) = (1/2\pi) \int_{-\pi}^{\pi} f(t) dt$$

$$\text{Ave}(f(t)+g(t)) = \text{Ave}(f(t)) + \text{Ave}(g(t))$$

$$\text{Ave}(\cos(nt)) = 0 \quad \text{for } n > 0 \quad \text{and} \quad \text{Ave}(\sin(nt)) = 0,$$

so applying Ave to (\*) :

$$\text{Ave}(f(t)) = a_0/2 \quad \text{or}$$

$$a_0 = (1/\pi) \int_{-\pi}^{\pi} f(t) dt.$$

Other integral expressions: This will use the trigonometric integrals [Slide]

$$\int_{-\pi}^{\pi} \cos(mt) \sin(nt) dt = 0$$

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mt) \cos(nt) dt &= 2\pi \quad \text{if } m = n = 0 \\ &= \pi \quad \text{if } m = n > 0 \\ &= 0 \quad \text{if } m \text{ is not equal to } n \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mt) \sin(nt) dt &= \pi \quad \text{if } m = n \\ &= 0 \quad \text{if } m \text{ is not equal to } n \end{aligned}$$

The first of these is easy, since the product is odd and the interval you are integrating over is symmetric. The others require some trig identity which you can find in Edwards and Penney.

Application: Substitute (\*) into  $\int_{-\pi}^{\pi} f(t) \cos(nt) dt$  (for  $n > 0$ )

Compute this integral term by term:

$$\int_{-\pi}^{\pi} (a_0/2) \cos(nt) dt = 0 \quad (\text{since } n > 0)$$

Then we have a bunch of cosines. The  $m$ -th one gives:

$$\int_{-\pi}^{\pi} a_m \cos(mt) \cos(nt) dt = a_m \pi \quad \text{if } m = n$$

$$= 0 \quad \text{if } m \text{ is not equal to } n$$

And then a bunch of sines. The  $m$ -th of them gives:

$$\int_{-\pi}^{\pi} a_m \sin(mt) \cos(nt) dt = 0$$

Only one of all these terms is nonzero: the cosine term with  $m = n$ , and since then  $a_m = a_n$ , we discover

$$\int_{-\pi}^{\pi} f(t) \cos(nt) dt = a_n \pi, \quad \text{or}$$

$$a_n = (1/\pi) \int_{-\pi}^{\pi} f(t) \cos(nt) dt$$

We did this calculation assuming  $n > 0$ , but since  $\cos(0t) = 1$  the formula is true for  $n = 0$  (by our comment about averages above).

Exactly the same method shows:

$$b_n = (1/\pi) \int_{-\pi}^{\pi} f(t) \sin(nt) dt$$

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