18.03 Class 30, Apr 21, 2010

Laplace Transform V: Poles and amplitude response

1. Recap on pole diagram
2. Stability
3. Exponential signals
4. Transfer and gain
5. Fourier and Laplace

[1] To recap: The kind of function F(s) that arises as a Laplace transform can be understood, in broad terms, by giving the set of points at which it becomes infinite. This is the "pole diagram."

Examples: The following F(s)'s all have pole diagram {2i,-2i} (a not zero) : [Slide]

F(s)	f(t)
a/(s^2+4)	(a/2) sin(2t)
as/(s^2+4)	a cos(2t)
e^{-bs}/(s^2+4)	cos_b(2t)
1 + a/(s^2+4)	delta(t) + (a/2) sin(2t)
4s/(s^2+4)^2	t cos(2t)

and many other examples. All these functions f(t) share some common features, for sufficiently large t :

- they oscillate with circular frequency 2 .
- they may grow or shink, but less than exponentially.

These features are common to all functions f(t) such that F(s) has this pole diagram.

Pole diagram {2i,-2i,1} : any of the above plus (c not zero)

F(s)	f(t)
c/(s-1)	c e^t + ...
c/(s-1)^2	c t e^t + ...

and many other examples. All these functions f(t) share some common features, for sufficiently large t :

- they grow as t ---> infinity "like e^t ." This means that they grow faster than e^{kt} for any k < 1, and slower than e^{lt} for any l > 1.
- they oscillate but the oscillations become insignificant relative to the size of f(t) as t ---> infinity.

The rightmost poles of F(s) determine the dominant features of f(t) for large t.

In particular: If all poles of $F(s)$ have negative real part - are to the left of the imaginary axis - then $f(t) \to 0$ as $t \to \infty$.

[2] This is a "stability" effect, related to one we have seen before. The unit impulse response $w(t)$ for $p(D)$ has Laplace transform $W(s) = 1/p(s)$. The poles of $W(s)$ are the roots of $p(s)$. If all roots have negative real part, then all solutions to $p(D)x = 0$ die off exponentially as $t \to \infty$. For $t > 0$, $w(t)$ is a solution to $p(D)x = 0$. If all the poles are in the left half plane, then not just the unit impulse response but in fact all solutions of the homogeneous equation die off: they are *transients.*

[3] Exponential signals

The reciprocal of the characteristic polynomial arose long ago, in the Exponential response formula:

Exponential response: $p(D) x = e^{rt}$ has an exponential solution

$$x_p = W(r) e^{rt}$$

The transfer function is the Laplace transform of the weight function.

This is just what appeared in our study of Sinusoidal response: [Slide]

$$p(D)x = \cos(\omega t)$$

$$p(D)z = e^{i \omega t}$$

$$z_p = W(i \omega) e^{i \omega t}$$

$$x_p = \mathrm{Re}[W(i \omega) e^{i \omega t}]$$

$$= \mathrm{gain}(\omega) \cos(\omega t - \phi)$$

where $$\mathrm{gain}(\omega) = |W(i \omega)|$$

$$- \phi = \mathrm{Arg}(W(i \omega))$$

$W(i \omega)$ is the "complex gain." (Here we are supposing that the input signal, with respect to which we should be measuring the gain and the phase lag, is just $\cos(\omega t)$.)

The gain is the restriction to the imaginary axis of the magnitude of the transfer function.

[4] Here's the vision that unifies most of what we have done in this course so far:

You have a system (a black box, with springs and masses and dashpots, for example) which you wish to understand. This means really that you want to be able to predict its response to various input signals.

We will only be able to analyze systems which are LINEAR and TIME

INVARIANT: so (1) superposition holds, and (2) delaying the input signal results in delaying the system response without otherwise changing it.

We are especially interested in its periodic response to periodic signals. Periodic signals decompose into sinusoidal signals, by Fourier series, so it's enough just to study sinusoidal system responses. There will be a gain and a phase lag involved. You'll be happy to understand the gain, and leave the phase lag for another day.

So hit the system: feed it delta(t) as input signal.

What comes out is w(t).

Apply L to w(t) to get W(s) .

Graph |W(s)| . This will be a surface lying over the complex plane. It probably has some poles.

Restrict s to purely imaginary values , s = i omega. This is what is needed to study sinusoidal input response:

p(D) x = e^{i omega t}

has exponential solution

x_p = W(i omega) e^{i omega t}

so |W(i omega)| is the gain.

The intersection of the graph of W(s) with the vertical plane lying over the imaginary axis is the amplitude response curve (extended to an even function, allowing negative omega).

Near resonance occurs because i omega is getting near to one of the poles of W(s) .

If you increase the damping, the poles move deeper into negative real part space, and eventually the two humps in the frequency response curve merge.

If you have a higher order system, you get more poles, and a more complicated amplitude response curve.

I did not talk about:

[5] Laplace and Fourier

A period function neither neither grows nor decays as t ---> infty, so we know that its rightmost poles lie in the imaginary axis.

A periodic function of period 2L has a Fourier series. Let's write omega = 2pi/2L = pi/L. Then the Fourier series has the form

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f(t) = a_0/2 + a_1 cos(omega t) + a_2 cos(2 omega t) + ...
             + b_1 sin(omega t) + b_2 sin(2 omega t) + ...
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The poles of L [sin(n omega t)] and of L [cos(n omega t)] occur at n omega i and - n omega i .

So the poles of u(t)f(t) occur along the imaginary axis, spaced out at multiples of +- i omega .

The location of poles of F(s) gives us some intuitive feeling for the significance of the s-domain. If there are poles to the right of the imaginary axis, then the corresponding function of time grows like e^{at} where a is the largest real part of any pole. The imaginary part of the pole tells us about the circular frequency of (exponentially damped or expanding) oscillations.

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