I. The winding number

Let \( R \subset \mathbb{R}^2 \) be an open region, let
\[
\begin{align*}
   x' &= F(x, y) \\
   y' &= G(x, y)
\end{align*}
\]
be an autonomous differential system on \( R \), and let \( C \subset R \) be an oriented, simpler closed curve in \( R \). In other words, \( C \) is the image of the circle under a 1-to-1 map whose derivative vector is always nonzero, say \( h : [0,1] \to R, h(1) = h(0) \).

If \( C \) contains no equilibrium point of the system, the following function is well-defined and continuous:
\[
f : C \to \delta^1 \subset \mathbb{R}^2, \quad f(q) = \frac{1}{\sqrt{F(q)^2 + G(q)^2}} \begin{bmatrix} F(q) \\ G(q) \end{bmatrix}.
\]

The composition \( foh : [0,1] \to \delta^1 \) is (essentially) a cts. map from the circle to the circle. To such a map there is an associated integer \( n \), the degree of the map. This integer counts the number of times \( foh(t) \) rotates counterclockwise around the circle as \( t \) rotates once counterclockwise around the circle. If \( h, F \) and \( G \) are all continuously differentiable function, \( g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \), and then the degree is simply
\[
\frac{1}{2\pi} \int_0^1 \left[ g_1(t)g_2'(t) - g_1'(t)g_2(t) \right] dt
\]

This integer turns out to be independent of \( h(t) \) (although it does depend on the orientation of \( C \)). it is called the winding number of \((F,G)\) about \( C \).

Let \( p \in R \) be an equilibrium point. It is isolated if there exists \( \varepsilon > 0 \) such that \( p \) is the only equilibrium point in the \( \varepsilon \)-ball about \( p \). For any \( 0 < q < \varepsilon \), consider the circle \( C_q \) of radius \( q \) centered at \( p \). The winding number of \((F,6)\) about \( C_q \) is independent of \( q \) and is called the index of \((F,6)\) at \( p \) (or sometimes the Poincare index).

**Examples:**

1. Let \( \lambda, \mu > 0 \) and let \( F = \lambda x, G = \mu y \). Then \( p = (0,0) \) is an isolated equilibrium point. Consider \( h_q(t) = \begin{bmatrix} q \cos(t) \\
   q \sin(t) \end{bmatrix} \), \( 0 < t \)

Then \( g(t) = \frac{1}{\sqrt{\lambda^2 \cos^2(t) + \mu^2 \sin^2(t)}} \begin{bmatrix} \lambda \cos(t) \\
   \mu \sin(t) \end{bmatrix} \).
And \( g_1(t)g_2'(t) - g_1'(t)g_2(t) = \frac{\lambda \mu}{\lambda^2 \cos^2(t) + \mu^2 \sin^2(t)} \). This is closely related to the Poisson kernel. It is nontrivial, but the integral \( \int_0^{2\pi} \frac{\lambda \mu}{\lambda^2 \cos^2(t) + \mu^2 \sin^2(t)} dt \) can be computed by elementary methods, and it equals \( 2\pi \) (consider the case that \( \lambda = \mu \)). So the index is +1.

(2) \( \lambda, \mu < 0 \). This is the same as above when \( \lambda \to -\lambda, \mu \to -\mu \). Notice the integral does not change. So the index is +1.

(3) \( \lambda < 0, \mu > 0 \). now the integral is \( -\int_0^{2\pi} \frac{(-\lambda)\mu}{(-\lambda)^2 \cos^2(t) + \mu^2 \sin^2(t)} dt \).

This is -1 times the integral from (1). So the index is -1.

**Theorem:** Let \( C \) be a simple closed curve that contains no equ. pts. in \( R \) oriented so the interior is always on the left. If the interior is contained in \( R \), and if the interior contains only finitely many equilibrium points, \( p_1, \ldots, p_n \), then the winding number about \( C \) is index \( (p_1) + \ldots + \text{index}(p_n) \). (and 0 if there are no eq. pts).

**Proof:** This is proved, for instance in Theorem 3, § 11.9 on p.442 of Wilfred Kaplan, *Ordinary Differential Equations*, Addison-Wesley, 1958.

**Corollary:** If \( R \) is simply-connected, then every cycle \( C \) contained in \( R \) contains an equilibrium point in its interior.

**Rmk:** A region \( R \) in \( \mathbb{R}^2 \) is simply-connected if for every simple closed curve \( C \) in \( R \), the interior of \( C \) is contained in \( R \). A cycle is a periodic orbit (that is necessarily a simple closed curve).

**Pf:** By construction, \( (F, 0) \) is parallel to the tangent vector of \( C \). Therefore the winding number is +1. So, by the theorem, there is an equilibrium point in the interior of \( C \).

II. Lyapunov functions

Let \( R \subset \mathbb{R}^n \) be an open region. Let \( \dot{x} = F(x) \) be an autonomous system on \( R \). Let \( p \in R \) be a point.

**Definition:** A function \( V : R \to \mathbb{R} \) is positive definite (resp. negative definite) if

1. \( V(q) \geq 0 \) (resp. \( V(q) \leq 0 \)) for all \( q \in R \)
2. \( V(q) = 0 \) iff \( q = p \).

Let \( p \) be an equilibrium point.
Definition: A strong Lyapunov function is a continuously differentiable function $V : \mathbb{R} \to \mathbb{R}$ such that

1. $V$ is positive definite
2. the function $V' := \sum_{i=1}^{n} \frac{dV(x)}{dx_i} F_i(x)$ is negative definite.

Remark: It is often the case that there is no strong Lyapunov function on $\mathbb{R}$, yet there is an open subregion $\mathbb{R}' \subset \mathbb{R}$ containing $p$ and a strong Lyapunov function on $\mathbb{R}'$. In this case, simply replace $\mathbb{R}$ by $\mathbb{R}'$ in what follows.

Hypothesis: Suppose a strong Lyapunov function exists. There is a minor issue that your book does not deal with: long-time existence of solution curves. Let $K \subset \mathbb{R}^n$ be a bounded closed region whose interior contains $p$ and such that $K \subset \mathbb{R}^n$. Define $r_0 = \text{minimum of } V$ on the bounded closed set $\partial K$ (a continuous function on a bounded closed subset of $\mathbb{R}^n$ always attains a minimum). Because $p \in$ interior of $K$, $r_0 > 0$. Define $\mathbb{R}'$ to be

$$\mathbb{R}' = KnV^{-1}\left([0, r_0]\right) = \left\{ q \in K \mid V(q) < r_0 \right\}.$$ 

Observe this is an open region in $\mathbb{R}$ that contains $p$ and is contained in the interior of $K$.

Theorem: (1) For every $x_0 \in \mathbb{R}'$, the solution curve $x(t)$ is defined for all $t > 0$.

(2) Moreover, $\lim_{t \to x} x(t) = p$. Therefore $p$ is an attractor and $\mathbb{R}'$ is in the basin of attraction of $p$.

Proof: For any $x_0 \in \mathbb{R}$, if $x(t)$ is defined on the interval $[0, t_1)$, consider $V(x(t))$ defined on $[0, t_1)$. By the Chain Rule, $V(x(t))$ is differentiable and

$$\frac{d}{dt} V(x(t)) = \sum_{i=1}^{n} \frac{\partial V(x(t))}{\partial x_i} \frac{dx_i(t)}{dt}.$$ 

By hypothesis, $x'_i(t) = F_i(x(t))$. Thus $\frac{d}{dt} V(x(t)) = V'(x(t))$.

By hypothesis, this is nonpositive. Therefore $V(x(t))$ is a non-increasing function. In particular, if $x_0 \in \mathbb{R}'$, then $x(t)$ is in $\mathbb{R}'$ for all $t \in [0, t_1)$.

(1) Let $x_0 \in \mathbb{R}'$. By way of contradiction, suppose that $x(t)$ is defined only on $[0, t_1)$ where $t_1$ is finite. By the theorem on maximally extended solutions, $\lim_{t \to t_1} x(t)$ exists and is in $\partial K$. Therefore $V(\lim_{t \to t_1} x(t)) \geq r_0$. Since $V$ is continuous

$$V(\lim_{t \to t_1} x(t)) = \lim_{t \to t_1} V(x(t)).$$ 

For all $t \geq 0$, $V(x(t)) \leq V(x_0) < r_0$.

So $\lim_{t \to t_1} V(x(t)) \leq V(x_0) < r_0$. This contradiction proves $x(t)$ is defined for all $t > 0$. 

18.034, Honors Differential Equations

Prof. Jason Starr
(2) Let $\varepsilon > 0$ and let $B_{\varepsilon}(p)$ denote the open ball of radius $\varepsilon$ centered at $p$. The set difference $K \setminus (KnB_{\varepsilon}(p))$ is closed and bounded. Therefore $V$ attains a minimum value $r_1$ on this set. Since $p$ is not in this set $r_1 > 0$. Also, 
$KnV^{-1}([r_1, \infty))$ is a closed set contained in $K$. So it is closed and bounded ($K$ is bounded). Therefore $V'$ attains a maximum value $-m_1$ on this set. Since $p$ is not in this set $-m_1 < 0$, i.e. $m_1 > 0$.

Define $t_1 = \frac{r_0 - r_1}{m_1}$.

The claim is that for all $x_0 \in R$, $V(x(t)) < r_1$ for all. In particular, since $x(t) \in R'$ & $V(x(t)) < r_1$, $x(t)$ is in $R' \cap B_{\varepsilon}(p)$. By way of contradiction, suppose $V(x(t)) \geq r_1$. By the mean value theorem, there exists $t'$ with $0 < t' < t$ such that $V(x_0) - V(x(t)) = V(x(t')) \cdot t$. Since $V(x(t)) \geq r_1$, also $V(x(t')) \geq r_1$.

Therefore $x(t') \in KnV^{-1}([r_1, \infty))$.

Thus $V'(x(t)) \geq m_1$. So $V(x_0) - V(x(t)) \geq m_1 t > m_1 t_1 = r_0 - r_1$. But $V(x_0) < r_0$ and $V(x(t)) \geq r_1$. This is a contradiction, proving $V(x(t)) < r_1$ for all $t > t_1$.

The definition of a weak Lyapunov function as well as the statements of Lyapunov's second and third theorems are in the textbook.

### III. A criterion for asymptotic stability.

Let $V$ be a real vector space of dimension $n$, eg. $\mathbb{R}^n$. Let $R \subset V$ be an open region, and let $\dot{x} = F(x)$ be an autonomous system on $R$. Let $p \in R$ be an equilibrium point.

**Theorem:** If $F$ is differentiable at $p$, and if every eigenvalue of $\left[ \frac{\partial F_i}{\partial x_j} \right]_p$ has negative real part, then there is an open region $R' \subset R$ contains $p$ and a story Lyapunov function on $R'$.

**Proof:** There is a beautiful proof in the first edition of the textbook, which is stapled at the end. Here we give a closely related, but different argument.

The Jacobian of $F$ at $p$ is a linear transformation $T : V \rightarrow V$ with the property that, for my norm $\| \cdot \|$ on $V$, for every $\varepsilon > 0$, $\exists \varepsilon > 0$ such that if $\|V\| < \varepsilon^2$, then $\left\| F(p) + \varepsilon^2 T \right\| < \varepsilon$. Notice this is independent of the system of coordinates on $V$. Without loss of generality, translate so $p = 0$.

As we have alluded to earlier in the semester, for each real vector space $V$ there is an associated complex vector space $V_c$ defined as a set to be $V \times \mathbb{C}$ with elements $(v, w)$ written $v + iw$. The addition is defined component-by-component. And for
each complex number \( \alpha + i\beta \), \((\alpha + i\beta) \cdot (\nu + i\nu)\) is defined to be \((\alpha\nu - \beta\nu) + i(\beta\nu + \alpha\nu)\).

The original vector space \( V \) is a subset by \( \nu \mapsto \nu + i\cdot 0 \). And \( T : V \to V \) extends to a \( \mathbb{C} \)-linear transformation \( T : V \to V \) by \( T(v + iw) = T(v) + iT(w) \).

By the Jordan normal form theorem, there exists a direct sum decomposition \( V \cong V_1 \oplus \cdots \oplus V_m \) and for each \( i = 1, \ldots, m \) an ordered basis \( B_i \) for \( V_i \) s.t.
(1) for each \( i = 1, \ldots, m \), \( T_i(V_i) \subseteq V_i \)
(2) the corresponding linear transformation \( T_i : V_i \to V_i \) has matrix

\[
[T_i]_{B_i,B_i} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}
\]

for some \( \lambda \).

For any nonzero \( \alpha \in \mathbb{C} \), there is also a basis \( B_{i,\alpha} \) s.t.

\[
[T_i]_{B_{i,\alpha},B_{i,\alpha}} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}.
\]

Indeed, if \( B_i = (v_1, \ldots, v_m) \), then \( B_{i,\alpha} = (v_1, \alpha v_2, \alpha^2 v_3, \ldots, \alpha^{m-1} v_m) \).

For each ordered basis \( B \) for \( V_i \), there is a “dual basis of coordinates” \( x_1, \ldots, x_m : V_i \to \mathbb{C} \) s.t.

\( V = x_1(v)v_1 + \ldots + x_m(v)\cdot v_m \) for every \( v \in V_i \) (\( B = (v_1, \ldots, v_m) \)). There is a corresponding Hermition inner product,

\[
\langle v, w \rangle_B : V_i \times V_i \to \mathbb{C}
\]

\[
\langle v, w \rangle_B = \sum_{i=1}^{m} x_i(v) \overline{x_i(w)}.
\]

In particular, this is bilinear, positive definite and \( \langle w, v \rangle_B = \langle v, w \rangle_B \).

And \( \langle T_i v, v \rangle_{B_{i,\alpha}} = \lambda|x_1|^2 + \alpha_1 x_2 \overline{x_2} + \lambda|x_2|^2 + \alpha_2 x_3 \overline{x_3} + \ldots + \alpha_{m-1} x_m \overline{x_m} + \lambda|x_m|^2 \).

**Lemma 1:** For each \( n \), the function on \( \mathbb{C}^n \),

\[
(x_1, \ldots, x_n) \mapsto |x_1|^2 - |x_1||x_2| + |x_2|^2 + \ldots + |x_k|^2 - |x_k||x_{k+1}| + |x_{k+1}|^2 + \ldots - |x_{n-1}||x_n| + |x_n|^2
\]

\[
= \sum_{k=1}^{n-1} |x_k|^2 - |x_k||x_{k+1}| + |x_{k+1}|^2
\]

is positive definite.

**Proof:** it is simply \( \frac{1}{2}|x_1|^2 + \frac{1}{2} \sum_{k=1}^{n-1} (|x_k| - |x_{k+1}|)^2 + \frac{1}{2}|x_n|^2 \).
Since it is a sum of squares, it is nonnegative. It is zero iff \(|x_1| = 0, |x_2| = |x_1| = 0, \ldots, |x_n| = |x_{n-1}| = 0\) and \(|x_n| = 0\), i.e. \(|x_1| = \ldots = |x_n| = 0\).

**Lemma 2**: If \(R_e(\lambda) < 0\) and if \(|\alpha| < -R_e(\lambda)\), then \(2R_e < T \nu, \nu >_{\beta, \alpha}\) is negative definite.

Moreover, \(2R_e < T \nu, \nu >_{\beta, \alpha} \leq 2(R_e(\lambda) + |\alpha|) \left( |x_1|^2 + \ldots + |x_n|^2 \right)\).

**Proof**: \(2R_e < T \nu, \nu >_{\beta, \alpha} = 2R_e(\lambda) \left( |x_1|^2 + \ldots + |x_n|^2 \right) + 2R_e(\alpha x_1 \overline{x_2} + \ldots + \alpha x_{n-1} \overline{x_n}) \leq 2R_e(\lambda) \left( |x_1|^2 + \ldots + |x_n|^2 \right) + 2|\alpha| \left( |x_1| |x_2| + \ldots + |x_{n-1}| |x_n| \right) = 2(R_e(\lambda) + |\alpha|) \left( |x_1|^2 + \ldots + |x_n|^2 \right) - |\alpha| \left( |x_1|^2 - |x_1| |x_2| + |x_2|^2 + \ldots + |x_{n-1}| |x_n| + |x_n|^2 \right) \)

By Lemma 1, \(-|\alpha| \left( |x_1|^2 - |x_1| |x_2| + |x_2|^2 + \ldots + |x_{n-1}| |x_n| + |x_n|^2 \right)\) is negative definite. Because \(R_e(\lambda) + \alpha < 0\), also \(2(R_e(\lambda) + |\alpha|) \left( |x_1|^2 + \ldots + |x_n|^2 \right)\) is negative definite.

For each \(i = 1, \ldots, n\), let \(R_e(\lambda) > \varepsilon_i > 0\). Let \(|x_i| + R_e(\lambda) < -\varepsilon_i\), i.e. \(0 < \varepsilon_i < |R_e(\lambda)| - \varepsilon_i\).

Define the function \(\|\|^2\) by \(\|v_1 + \ldots + v_n\|^2 = \sum_{i=1}^{n} <v_i, v_i>_{B_i, \alpha}\) where \(v_i \in V_i\).

This is a positive definite function. Moreover, \(\|\| := \sqrt{\|\|^2}\) is a norm. Therefore there is a \(\delta > 0\) s.t. if \(\|v\| < \delta\), then \(F(v) - T \nu \leq \min (\varepsilon_1, \ldots, \varepsilon_n) \|v\|\).

Now \(\frac{d}{dt} \|x(t)\|^2 = 2R_e < F(x), x(t) > = \sum_{i=1}^{n} 2R_e < T_i x, x > + 2R_e < F(x) T \nu, \nu >\)

So \(2R_e < F(x), x > \leq \sum_{i=1}^{n} 2R_e < T_i x, x > + \left( F(x) - T_x \right) \|x\| \).

By Lemma 2, this is \(\leq -2 \min (\varepsilon_1, \ldots, \varepsilon_n) \|x\|^2 + \left( F(x) - T_x \right) \|x\|\)

If \(\|x\| < \delta\), this is \(\leq -2 \min (\varepsilon_1, \ldots, \varepsilon_n) \|x\|^2 + \min (\varepsilon_1, \ldots, \varepsilon_n) \|x\|^2 = -\min (\varepsilon_1, \ldots, \varepsilon_n) \|x\|^2\)

So this is negative semidefinite. Therefore \(\|\|^2\) is a strong Lyapunov function on the ball of radius \(R\) centered at \(p\).