18.034 Problem Set #2

Due by Friday, February 20, 2009, by NOON.

Notation. $' = d/dx$.

1. Let
   
   \[ f(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0, \\ k & \text{for } x = 0, \end{cases} \]

   where $k$ is a constant. Show that no matter how the constant $k$ is chosen, the differential equation $y' = f(x)$ has no solution on an interval containing the origin.

2. Suppose that $f$ be a continuous bounded function for the entire real axis. If $f'$ is continuous, then show that the nonzero solution of the initial value problem of $y' = yf(y)$ with $y(0) = y_0 \neq 0$ exists for all $x$. (You may need to assume the uniqueness theorem.)


4. (The Ricatti equation) It is the differential equation of the form $y' = a(x) + b(x)y + c(x)y^2$. In general the Ricatti equation is not solvable by elementary means*. However,
   (a) show that if $y_1(x)$ is a solution then the general solution is $y = y_1 + u$, where $u$ is the general solution of a certain Bernoulli equation (cf. pset #1).

   (b) Solve the Ricatti equation $y' = 1 - x^2 + y^2$ by the above method.

5. Let

   \[ Ly = y'' + y. \]

   We are going to find the rest solution of the differential equation $Ly = 3 \sin 2x + 3 + 4e^x$. That is the solution with $u(0) = u'(0) = 0$.
   (a) Find the general solution of $Ly = 0$.
   (b) Solve $Ly = 3 \sin 2x$, $Ly = 3$, and $Ly = 4e^x$ by use of appropriate trial solutions.
   (c) Determine the constants in

   \[ y(x) = c_1 \cos x + c_2 \sin x - \sin 2x + 3 + 2e^x \]

   to find the solution.

6. (Euler’s equi-dimensional equation) It is a differential equation of the form $x^2y'' + pxy' + qy = 0$, where $p, q$ are constants.
   (a) Show that the setting $x = e^t$ changes the differential equation into an equation with constant coefficients.
   (b) Use this to find the general solution to $x^2y'' + xy' + y = 0$.
   (c) For which values of $p$, the general solutions of $x^2y'' + pxy' + 2y = 0$ are defined for the entire real axis $(-\infty, \infty)$?

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*This was shown by Liouville in 1841.