18.034 Problem Set #5  
(modified on March 30, 2009)

Due by Friday, April 3, 2009, by NOON.

1. (a) Let \( f_n(t), n = 1, 2, \ldots \) be continuous functions on an interval \([a, b]\) and \( \{f_n(t)\} \) converge uniformly to \( f(t) \) on \([a, b]\). Show that

\[
\lim_{n \to \infty} \int_a^b f_n(t) \, dt = \int_a^b f(t) \, dt.
\]

(b) Construct \( \{f_n(t)\} \) on \([0, 1]\) such that the above equality does not hold true.

2. For the initial value problem \( x' = f(t, x) \) with \( x(t_0) = x_0 \), where \( f \) is continuous and Lipschitzian in the rectangle \([t - t_0] \leq T \) and \([x - x_0] \leq K \) with the Lipschitzian constant \( L \), suppose the exact solution \( x \) and the Picard iterates \( x_n \) all exist over one and the same interval of \( t \). Show that on such an interval

\[
|x(t) - x_n(t)| \leq ML^n \frac{T^{n+1}}{(n + 1)!} e^{LT},
\]

where \( |f(t, x)| \leq M \) in \([t - t_0] \leq T \) and \([x - x_0] \leq K \).

3. Let \( f \) be a real-valued continuous function in the rectangle \([t - t_0] \leq T \) and \([x - x_0] \leq K \). Consider the initial value problem

\[ (1) \quad x'' = f(t, x), \quad x(t_0) = x_0, \quad x'(t_0) = x_1. \]

(a) Show that \( \phi \) is a solution of (1) if and only if \( \phi \) is a solution of the integral equation

\[ (2) \quad x(t) = x_0 + (t - t_0)x_1 + \int_{t_0}^t (t - s)f(s, x) \, ds. \]

(b) Let \( \{x_n\} \) be a successive approximation for (2). That is, \( x_0(t) = x_0 \) and

\[
x_n(t) = x_0 + (t - t_0)x_1 + \int_{t_0}^t (t - s)f(s, x_{n-1}) \, ds \quad n = 1, 2, \ldots.
\]

If \( f(t, x) \) is continuous and Lipschitzian with respect to \( x \) in the rectangle \([t - t_0] \leq T \) and \([x - x_0] \leq K \), show that \( \{x_n\} \) converges on the interval \([t - t_0] \leq \min(T, K/B) \) to the solution of (1), where \( B = |x_1| + MT/2 \) and \( |f(t, x)| \leq M \) in \([t - t_0] \leq T \) and \([x - x_0] \leq K \).

4. (a) Show that \( \mathcal{L}(1/\sqrt{t}) (s) = \sqrt{\pi/s} \) by using the well-known formula

\[
\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]

(b) Use part (a) to show that \( \mathcal{L}(\sqrt{t}) (s) = \frac{\sqrt{\pi}}{2s^{3/2}} \) for \( s > 0 \).

5. (a) Show that \( \mathcal{L}(\exp(t^2)) (s) \) does not exist for any interval of the form \( s > a \).

(b) For what values of \( k \), will \( \mathcal{L}(1/t^k) \) exist?

6. Find the functions whose Laplace transforms are the following functions:

(a) \[ \frac{5s - 6}{s^2 + 4} \quad \text{and} \quad \frac{2}{s^2 + 4}; \]

(b) \[ \frac{9s + 3}{9s^2 + 6s + 19}. \]

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*"A mathematician is one to whom that is as obvious as that twice two makes four is to you. Liouville was a mathematician." — Lord Kelvin*