Problem set 9, Solution keys

   Folia of Descartes are in figure 5.2.

2. (a) Suppose not. This means at least one of the inequalities \( f > 0, f < 0, g > 0, g < 0 \) holds at \((x_1, y_1)\). Without loss of generality, let \( f(x_1, y_1) = -2\alpha < 0 \). (other cases are similar).
   By continuity, \( f(x(t), y(t)) < -\alpha \) for \( t > T \) for some large \( T > 0 \).
   Hence, \( x'(t) \leq -\alpha \) for \( t > T \), and \( x(t) \leq -\alpha t + \beta \) for some \( \beta \) for \( t > T \).
   Then \( x(t) \to -\infty \) as \( t \to \infty \), which contradicts that \( x(t) \to x_1 \).

(b) Without loss of generality, let \( x_0 = y_0 = 0 \).
   Let \( F(t) = f(xt, yt) \) in a disk \( r < \delta \) for some \( \delta > 0 \). Hence \( r = \sqrt{x^2 + y^2} \).
   By the Mean Value Theorem, \( F(1) - F(0) = F'(\tau) \) for some \( 0 < \tau < 1 \).
   And, \( F'(\tau) = f_x(x\tau, y\tau)x + f_y(x\tau, y\tau)y \).
   Since \( f_x \) and \( f_y \) are continuous, \(|f_x(x\tau, y\tau) - f_x(0, 0)| < \epsilon(r), |f_y(x\tau, y\tau) - f_y(0, 0)| < \epsilon(r)\)
   and \( \epsilon(r) \to 0 \) as \( r \to 0 \).
   Therefore \( a \to f_x(0, 0), \ b \to f_y(0, 0) \), as \( r \to 0 \). A similar argument applies to \( g \).

3. \((0, 0)\) unstable singular node
   \((\frac{3}{2}, 0), (0, \frac{3}{2})\) saddles.
   \((1, 1)\) stable nodes.

4. \((0, 0), (0, 3)\) saddles.
   \((1, 2)\) stable focus.
   
   (b) Let $E(x) = x^2$ a Lyapunov function. $E(0) = 0$ and $E(x) > 0$ for $x \neq 0$.
   
   And $\dot{E}(x) = 2xf(x) < 0$ for $x \neq 0$ and $x$ near 0 since $f'(0) < 0$.
   
   Therefore 0 is asymptotically stable.

6. (a) In addition to $x = 0$, $x = \frac{n\pi}{n+1}$, $n = 1, 2, 3, \ldots$ are critical points. Let $0 < |x(0)| \ll 1$ so that $|x(t)| < \frac{1}{n+1}$ for some $n$ large.
   
   Then $|x(t)| < \frac{n+1}{n} |x(0)| \leq \frac{n}{n+1}$ for all $t$ since a non-stationary solution cannot pass through a critical point.
   
   Therefore $|x(t)| < \frac{n+1}{n} |x(0)| \leq 2|x(0)|$, and 0 is stable.

   Take $x_n(0) = \frac{1}{n\pi}, n = 1, 2, 3, \ldots$. Then $x_n(t) = \frac{1}{n\pi}$ for all $t$. So, 0 is not asymptotically stable.

   (b) The linear system is
   \[
   \begin{pmatrix}
   x \\
   y
   \end{pmatrix}' = \begin{pmatrix}
   0 & 1 \\
   0 & 0
   \end{pmatrix} \begin{pmatrix}
   x \\
   y
   \end{pmatrix}
   \]
   
   The solutions are $x(t) = a + bt, y(t) = b$. Therefore (0, 0) is unstable.

   For the nonlinear system, let $E(x, y) = x^4 + 2y^2$. Then, $E(0, 0) = 0, E(x, y) > 0$ for $(x, y) \neq (0, 0)$.
   
   $\dot{E}(x, y) = 4x^3(y - x^3) + 4y(-x^3) = -4x^6 \leq 0$. So, (0, 0) is stable.