Lecture 22. Thursday April 30: Dirchlet problem continued

I did not finish the proof last time:-

*Proof.* Notice the form of the solution in case $V \geq 0$ in (21.25). In general, we can choose a constant $c$ such that $V + c \geq 0$. Then the equation

\begin{equation}
-\frac{d^2w}{dx^2} + Vw = Tw_k \iff -\frac{d^2w}{dx^2} + (V + c)w = (T + c)w.
\end{equation}

Thus, if $w$ satisfies this eigen-equation then it also satisfies

\begin{equation}
 w = (T + c)A(\text{Id} + A(V + c)A)^{-1}Aw \iff Sw = (T + c)^{-1}w, \quad S = A(\text{Id} + A(V + c)A)^{-1}A.
\end{equation}

Now, we have shown that $S$ is a compact self-adjoint operator on $L^2(0,2\pi)$ so we know that it has a complete set of eigenfunctions, $e_k$, with eigenvalues $\tau_k \neq 0$. From the discussion above we then know that each $e_k$ is actually continuous – since it is $Aw'$ with $w' \in L^2(0,2\pi)$ and hence also twice continuously differentiable. So indeed, these $e_k$ satisfy the eigenvalue problem (with Dirichlet boundary conditions) with eigenvalues

\begin{equation}
 T_k = \tau_k^{-1} + c \to \infty \text{ as } k \to \infty.
\end{equation}

The solvability part also follows much the same way. \hfill \Box