I. Consider the Morawetz vectorfield $\overline{K}^\mu$ on $R^{1+3}$ defined by

\begin{align}
0.0.1 & \quad \overline{K}^0 = 1 + t^2 + (x_1)^2 + (x_2)^2 + (x_3)^2, \\
0.0.2 & \quad \overline{K}^j = 2tx^j, \quad (j = 1, 2, 3).
\end{align}

\textbf{a)} Show that $\overline{K}$ is future-directed and timelike. Above, $(t, x^1, x^2, x^3)$ are the standard coordinates on $R^{1+3}$.

\textbf{b)} Show that

\begin{equation}
0.0.3 \quad \partial_\mu \overline{K}_\nu + \partial_\nu \overline{K}_\mu = 4tm_{\mu\nu}, \quad (\mu, \nu = 0, 1, 2, 3),
\end{equation}

where $m_{\mu\nu}$ denotes the Minkowski metric.

\textbf{Remark 0.0.1.} $\overline{K}$ is said to be a \textit{conformal Killing field} of the Minkowski metric because the right-hand side of (0.0.3) is proportional to $m_{\mu\nu}$.

c) Show that

\begin{equation}
0.0.4 \quad m_{\mu\nu}T^{\mu\nu} = -(m^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi,
\end{equation}

where $T^{\mu\nu} \equiv \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}m_{\mu\nu}(m^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$ is the energy-momentum tensor corresponding to the linear wave equation, and $T^{\mu\nu} \equiv (m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}T_{\alpha\beta}$ is the energy-momentum tensor with its indices raised.

d) Show that $\partial_\mu (\overline{K}) J^\mu = 2t(m^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$ whenever $\phi$ is a $C^2$ solution to the linear wave equation $(m^{-1})^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$, where

\begin{equation}
0.0.5 \quad (\overline{K}) J^\mu \equiv -T^{\mu\nu} \overline{K}_\nu.
\end{equation}

e) Show that $\partial_\mu \tilde{J}^\mu = 0$ whenever $\phi$ is a $C^2$ solution to the linear wave equation $(m^{-1})^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$, where

\begin{equation}
0.0.6 \quad \tilde{J}^\mu \equiv (\overline{K}) J^\mu - 2t\phi(m^{-1})^{\mu\alpha} \partial_\alpha \phi + \phi^2(m^{-1})^{\mu\alpha} \partial_\alpha t.
\end{equation}

f) Show that

\begin{equation}
0.0.7 \quad (\overline{K}) J^0 = \frac{1}{4} \left\{ [1 + (t + r)^2](\nabla_L \phi)^2 + [1 + (t - r)^2](\nabla_L \phi)^2 + 2[1 + t^2 + r^2] \right\} \phi n^{\mu\nu} \partial_\mu \phi \partial_\nu \phi.
\end{equation}
Above, \((m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu L^\nu - \frac{1}{2}L^\mu L^\nu + f^{\mu\nu}\) is the standard null decomposition of \((m^{-1})^{\mu\nu}\) from class. In particular, \(L^\mu = (1, \frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}), L^\mu = (1, -\frac{x^1}{r}, -\frac{x^2}{r}, -\frac{x^3}{r}), \nabla_L \phi = \partial_t \phi + \partial_r \phi, \nabla_L \phi = \partial_t \phi - \partial_r \phi,\) and \(f^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\) is the square of the Euclidean norm of the angular derivatives of \(\phi.\) Here, \(r \overset{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\) denotes the standard spherical coordinate on \(\mathbb{R}^3,\) and \(\partial_r\) denotes the standard radial derivative.

**Hint:** The following expansions in terms of \(L\) and \(L\) may be very helpful:

\[(0.0.8)\]
\[
\overline{K}^\mu = \frac{1}{2}\{[1 + (t + r)^2]L^\mu + [1 + (t - r)^2]L^\mu]\}
\[(0.0.9)\]
\[
(1, 0, 0, 0) = \frac{1}{2}(L^\mu + L^\nu),
\]
\[(0.0.10)\]
\[
(\overline{K}) J^0 = T(\overline{K}, \frac{1}{2}(L + \overline{L}))
\]
\[
= \frac{1}{4}\{[1 + (r + t)^2]T(L, L) + [1 + (t - r)^2]T(L, \overline{L}) + ([1 + (r + t)^2] + [1 + (r - t)^2])T(L, \overline{L})\).
\]

**f)** Show that

\[(0.0.11)\]
\[
\overline{J}^0 = \frac{1}{4}\{[1 + (t + r)^2](\nabla_L \phi)^2 + [1 + (t - r)^2](\nabla_L \phi)^2 + 2[1 + t^2 + r^2] f^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\} + 2t \phi \partial_t \phi - \phi^2.
\]

**g)** Show that

\[(0.0.12)\]
\[
- \int_{\mathbb{R}^3} \phi^2 d^3 x = \frac{2}{3} \int_{\mathbb{R}^3} r \phi \partial_r \phi d^3 x
\]
whenever \(\phi\) is a \(C^1,\) compactly supported function.

**Hint:** Use the identity \(1 = \partial_r r\) together with integration by parts in spherical coordinates and the fact that \(d^3 x = r^2 \sin \theta d\theta d\phi d\phi\) in spherical coordinates.

**h)** Use parts f) and g) to show that

\[(0.0.13)\]
\[
\overline{J}^0 = \frac{1}{4}\{(\nabla_L \phi)^2 + (\nabla_L [(t + r) \phi])^2 + (\nabla_L \phi)^2 + (\nabla_L [(t - r) \phi])^2 + 2[1 + t^2 + r^2] f^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\}.
\]

**i)** Finally, with the help of the vectorfield \(\overline{J}^\mu,\) part e), and part h), apply the divergence theorem on an appropriately chosen spacetime region to derive the following conservation law for smooth solutions to the linear wave equation \((m^{-1})^{\mu\nu} \partial_\mu \partial_\nu \phi = 0 :\)
\begin{equation}
\frac{1}{4} \int_{\mathbb{R}^3} \left\{ (\nabla_L \phi)^2 + \left( \nabla_L \left[ (t + r) \phi \right] \right)^2 + (\nabla_L \phi)^2 + \left( \nabla_L \left[ (t - r) \phi \right] \right)^2 + 2[1 + t^2 + r^2] \mu_{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} \, d^3 x
= \frac{1}{4} \int_{\mathbb{R}^3} \left\{ (\nabla_L \phi)^2 + \left( \nabla_L \left[ (t + r) \phi \right] \right)^2 + (\nabla_L \phi)^2 + \left( \nabla_L \left[ (t - r) \phi \right] \right)^2 + 2[1 + t^2 + r^2] \mu_{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} \, d^3 x \bigg|_{t=0},
\end{equation}

where the left-hand side is evaluated at time \( t \), and right-hand side is evaluated at time \( t = 0 \). For simplicity, at each fixed \( t \), you may assume that there exists an \( R > 0 \) such that \( \phi(t, x) \) vanishes whenever \( |x| \geq R \).

**Remark 0.0.2.** Note that the right-hand side of (0.0.14) can be computed in terms of the initial data alone. Note also that the different null derivatives of \( \phi \) appearing on the left-hand side of (0.0.14) carry different weights. In particular, \( \nabla_L \phi \) and the angular derivatives of \( \phi \) have larger weights than \( \nabla_L \phi \). These larger weights are strongly connected to the following fact, whose full proof requires additional methods going beyond this course: \( \nabla_L \phi \) and the angular derivatives of \( \phi \) decay faster in \( t \) compared to \( \nabla_L \phi \).
18.152 Introduction to Partial Differential Equations.
Fall 2011

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.