Lecture 8: Cylindrical separability - Bessel functions

\[ \Omega = \text{interior of circle in } \mathbb{R}^2 \]

\[ \hat{A} = \nabla^2 : \hat{A} = \hat{A}^* , \text{ negative definite } \Rightarrow \text{real } \lambda < 0 , \perp \text{ eigenfunctions} \]

separation ansatz: \[ \nabla^2 u = \lambda u \Rightarrow \text{separable } u(r, \theta) = \rho(r) \tau(\theta) \]

\[ \Rightarrow \nabla^2 u = \left[ \frac{1}{r} \left( \frac{d}{dr} r \frac{d}{dr} \right) + \frac{1}{r} \frac{d^2}{d\theta^2} \right] u = \frac{1}{r} (r \rho')' \tau + \frac{1}{r} r \rho \tau'' = \lambda \rho \tau \]

\[ \Rightarrow \frac{r (r \rho')'}{\rho} - r^2 \lambda = -\frac{\tau''}{\tau} = \# = +m^2 \]

\[ \begin{aligned} &\text{r only} \\ &\text{\theta only} \end{aligned} \]

\[ \Rightarrow \frac{\tau''}{\tau} = -\# \Rightarrow \tau(\theta) = \sin(m \theta) \quad \text{or } \cos(m \theta) \text{ (or exp?)} \quad \text{(or any linear comb.)} \]

\[ \Rightarrow \begin{cases} \tau(\theta) = \cos(m \theta) \quad \text{periodic: } \tau(\theta + 2\pi) = \tau(\theta) \Rightarrow \# = m \text{ integer} \\ \tau(\theta) = \sin(m \theta) \quad \text{or } \sin(m \theta) \end{cases} \]

\[ \Rightarrow r (r \rho')' = (r^2 \lambda + m^2) \rho = 0 \]

\[ \begin{aligned} &\lambda < 0 \Rightarrow \text{let } \lambda = -k^2 / \text{for some } k \\ &r^2 \rho'' + \rho' + (k^2 r^2 - m^2) \rho = 0 \end{aligned} \]

let \[ z = kr \Rightarrow \frac{d^2 \rho}{dz^2} + \frac{3}{z} \frac{d\rho}{dz} + (k^2 - m^2) \rho = 0 \]

"Bessel's equation" of order \( m \)
2) solutions must be some functions \( J_m (\bar{r}) = \frac{J_m (kr)}{\bar{r}} = \rho (r) \)

where \( J_m \) is "Bessel function of 1st kind"

= "cylindrical analogue" of sine/cosine
  - standard functions, built into Matlab etc.

why oscillating? consider large \( r \):

\[
0 = r^2 \rho'' + r \rho' + (k^2 r^2 - m^2) \rho \approx r^2 (\rho'' + k^2 \rho)
\]

\[\Rightarrow \rho (r) \propto \sin \text{ or } \cos \text{ of } kr\]

a little more carefully: suppose \( \rho (r) \propto \cos (kr) \cdot r^p \) (or \( \sin \))

\( kr \gg 1 \) for some unknown power \( p \)
\[ 0 = r^2 \rho'' + r \rho' + (k^2 r^2 - m^2) \rho \]

\[ r^{p+2} \rho \\
\approx -k^2 r^2 \cos(kr) \rho^p \\
-2kr^2 \sin(kr) \rho^{p-1} \rho \\
+ k^2 r^2 \cos(kr) \rho^p \\
+ r^2 \cos(kr) \rho^{p-2} \rho (p-1) \\
+ r \cos(kr) \rho^{p-1} \rho \\
- m^2 \cos kr \rho \]

\[ k r \gg 1 \Rightarrow \rho = -\frac{1}{2} \]

\[ \rho(r) \approx \frac{\cos or \sin of kr}{\sqrt{r}} \quad (\times some\ normalization) \]

Fancier analysis \[ \Rightarrow \ldots \Rightarrow J_m(kr) \approx \frac{2}{\sqrt{\pi kr}} \cos(kr - \frac{m\pi - \pi}{2}) \]

\[ k r \gg m^2 \]

* Eigenvalues:

\[ \rho(R) = 0 = J_m(kR) \]

\[ \Rightarrow kR \text{ is root of } J_m \]

Let n\textsuperscript{th} root of \( J_m(\beta) \) = \( \beta_{m,n} \)

\[ \Rightarrow k_{m,n} = \frac{\beta_{m,n}}{R} \]

\[ \Rightarrow \lambda = -\left(\frac{\beta_{m,n}}{R}\right)^2 \]

[Compare to 1d : \( \sin(kx) \) solutions, \( \sin(kL) = 0 \)

\( \Rightarrow kL \) is root of \( \sin() = n\pi \) ]
orthogonality: if \( u_m,n \) and \( u_{m',n'} \) are eigenfunctions with \( \lambda_{mn} \neq \lambda_{m'n'} \)

\[
\Rightarrow \langle u_m,n | u_{m',n'} \rangle = 0
\]

\[
\Rightarrow \int_0^{2\pi} d\theta \cos(m\theta) \cos(m'\theta) \int_0^R r \ dr \ J_m(k_{mn}r) J_m(k_{m'n'}r)
\]

must be 0 if \( m \neq m' \) \( n \neq n' \)

\[
\Rightarrow \int_0^R r \ dr \ J_m(k_{mn}r) J_m(k_{m'n'}r)
\]

let \( x = k_{mn}r \)

\[
= R^2 \int_0^{\frac{x}{2\pi}} dx \ J_m(\beta_{mn}x) J_m(\beta_{m'n'}x) = 0
\]

for \( n \neq n' \) (\( \Rightarrow \) must be oscillating!)
**Small- \( r \) behavior and the missing Bessel solution:**

- Bessel's equation is 2\(^{nd}\) order \( \left( \frac{d^2}{dr^2} \right) \) \( \Rightarrow \) has 2 index, sols!
- Consider behavior for \( kr << 1 \), suppose \( \rho(r) \sim r^p \)
  - for small \( r \)
  - for some unknown power \( p \)

\[ 0 = r^2 \rho'' + r \rho' + (k^2 r^2 - m^2) \rho = \rho (p-1) r^p + p r^p + k^2 r^{p+2} - m^2 r^p \]

\[ \approx r^p \left[ p (p-1) + p - m^2 \right] \]

\[ = r^p (p^2 - m^2) \]

\[ \Rightarrow p = \pm m \Rightarrow \text{two possible solutions:} \]

1\(^{st}\) kind: \( J_m(kr) \sim r^m \) for small \( kr \)

2\(^{nd}\) kind: \( Y_m(kr) \sim r^{-m} \) for small \( kr \)

\[ m = 0 \text{ case is trickier: } Y_0(kr) \sim \log(r) \]

**Here, \( Y_m \) is not an allowed eigenfunction**

since we require \( \text{finite solutions at } r \to 0 \)

\[ \Rightarrow \text{eigenfunctions are:} \]

\[ J_m(k_m r) \cos(m \theta) \text{ and } J_m(k_m r) \sin(m \theta) \]

\[ \text{for } \lambda = -k_m^2, \quad k_m = \frac{3m}{R} \]