Lecture 17

Derived Green's function of $\nabla^2$ in 3d for infinite space (requiring solutions to $\to$ zero at infinity to get a unique solution), in three steps:

1. Because the $\nabla^2$ operator is invariant under translations (changes of variables $x \to x+y$), showed that $G(x,x')$ can be written as $G(x,x')=G(x-x',0)$. Similarly, rotational invariance implies that $G(x-x',0)=g(|x-x'|)$ for some function $g(r)$ that only depends on the distance from $x'$.

2. In spherical coordinates, solved $-\nabla^2 g = 0$ for $r > 0$ (away from the delta function), obtaining $g(r)=c/r$ for some constant $c$ to be determined.

3. Took the distributional derivative $(-\nabla^2 g)\{\varphi\}=g\{-\nabla^2 \varphi\}$ ("integrating by parts" using the fact from Lecture 7 that $\nabla^2$ is self-adjoint) for an arbitrary test function $\varphi(x)$, and showed by explicit integration that we get $c\varphi(0)$. Therefore $c=1/4\pi$ for us to solve $-\nabla^2 g = \delta(x-x')$.

Hence $G(x,x') = 1/4\pi|x-x'|$ for this problem, and $-\nabla^2 u=f$ is solved by $u(x)=\int f(x')d^3x'/4\pi|x-x'|$.

A physical example of this can be found in electrostatics, from 8.02: the potential $V$ of a charge density $\rho$, satisfies $-\nabla^2 V=\rho/\varepsilon_0$. A point charge $q$ at $x'$ is a charge density that is zero everywhere except for $x'$, and has integral $q$, hence is $\rho(x)=q\delta(x-x')$. Solving for $V$ is exactly our Green's function equation except that we multiply by $q/\varepsilon_0$, and hence the solution is $V(x)=\rho(4\pi\varepsilon_0|x-x'|)$, which should be familiar from 8.02. Hence $-\nabla^2 V=\rho/\varepsilon_0$ is solved by $V(x)=\int \rho(x')d^3x'/4\pi\varepsilon_0|x-x'|$, referred to in 8.02 as a "superposition" principle (writing any charge distribution as the sum of a bunch of point charges).

Perhaps the most important reason to solve for $G(x,x')$ in empty space is that solutions for more complicated systems, with boundaries, are "built out of" this one.

An illustrative example is $\Omega$ given by the 3d half-space $z>0$, with Dirichlet boundaries (solutions=0 at $z=0$). For a point $x'$ in $\Omega$, showed that the Green's function $G(x,x')$ of $-\nabla^2$ is $G(x,x')=(1/|x-x'| - 1/|x-x''|)/4\pi$, where $x''$ is the same as $x'$ but with the sign of the $z$ component flipped. That is, the solution in the upper half-space $z>0$ looks like the solution from two point sources $\delta(x-x')-\delta(x-x'')$, where the second source is a "negative image" source in $z<0$. This is called the method of images.

Reviewed method-of-images solution for half-space. There are a couple of other special geometries where a method-of-images gives a simple analytical solution, but it is not a very general method (complicated generalizations for 2d problems notwithstanding). The reason we are covering it, instead, is that it gives an analytically solvable example of a principle that is general: Green's functions (and other solutions) in complicated domains look like solutions in the unbounded domain plus extra sources on the boundaries.

Further reading: See e.g. sections 9.5.6–9.5.8 of Elementary Applied Partial Differential Equations by Haberman for a traditional textbook treatment of Green's functions of $\nabla^2$ in empty space and the half-space. If you Google "method of images" you will find lots of links, mostly from the electrostatics viewpoint see also e.g. Introduction to Electrodynamics by Griffiths for a standard textbook treatment; the only mathematical difference introduced by (vacuum) electrostatics is the multiplication by the physical constant $\varepsilon_0$ (and the identification of $-\nabla V$ as the electric field).