1 Problem 1

Do problem 10.4.3 in Haberman (p 469). The answer for (a) is in the back - please show how to get that answer. After doing parts (a), (b), solve the same PDE on the semi-infinite rod \( \{ x \geq 0 \} \) with an insulated BC at \( x = 0 \):

\[
\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = 0
\]

and the IC

\[
u (x, 0) = \delta (x - 1), \quad x > 0.
\]

We also assume \( u \) is bounded as \( x \to \infty \).

**Solutions:** (a) The problem 10.4.3 is to solve the diffusion equation with convection,

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + c \frac{\partial u}{\partial x}, \quad -\infty < x < \infty, \quad t > 0,
\]

\[
u (x, 0) = f (x), \quad -\infty < x < \infty.
\]

Define the Fourier Transform as

\[
\mathcal{F} [u (x, t)] = \hat{U} (\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u (x, t) e^{i\omega x} dx
\]

Taking the Fourier Transform of the PDE gives, from our rules in class,

\[
\frac{\partial}{\partial t} \hat{U} (\omega, t) = -k \omega^2 \hat{U} (\omega, t) - ci\omega \hat{U} (\omega, t) = (-k \omega^2 - ci\omega) \hat{U} (\omega, t)
\]

Integrating gives

\[
\hat{U} (\omega, t) = C (\omega) e^{-k\omega^2 t - ci\omega t}
\]
Imposing the IC gives
\[ C(\omega) = \bar{U}(\omega, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, 0) e^{i\omega x} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx. \]

Thus \( C(\omega) = F(\omega) \) is the Fourier Transform of \( f(x) \). Lastly,
\[ \bar{U}(\omega, t) = F(\omega) e^{-k\omega^2 t} e^{-ci\omega t} \]

Note the inverse FT’s:
\[ \mathcal{F}^{-1}[F(\omega)] = f(x), \quad \mathcal{F}^{-1}[e^{-k\omega^2 t}] = \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt} \]

To find the inverse FT, we use the convolution theorem to obtain, as in class,
\[ \mathcal{F}^{-1}[F(\omega) e^{-k\omega^2 t}] = \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi kt}} \exp \left( -\frac{(x-s)^2}{4kt} \right) ds \]

We now use the Shifting Theorem (Table on p 468),
\[ \mathcal{F}^{-1}[e^{-i\omega \beta} G(\omega)] = \int_{-\infty}^{\infty} e^{-i\omega \beta} G(\omega) e^{-i\omega x} d\omega \]
\[ = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega(\beta + x)} d\omega \]
\[ = g(x + \beta) \]

so that
\[ u(x, t) = \mathcal{F}^{-1}[\bar{U}(\omega, t)] \]
\[ = \mathcal{F}^{-1}[e^{-ci\omega t} F(\omega) e^{-k\omega^2 t}] \]
\[ = \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi kt}} \exp \left( -\frac{(x+ct-s)^2}{4kt} \right) ds \]

(b) Consider the IC \( f(x) = \delta(x) \). Substituting \( f(s) = \delta(s) \) gives
\[ u(x, t) = \int_{-\infty}^{\infty} \frac{\delta(s)}{\sqrt{4\pi kt}} \exp \left( -\frac{(x+ct-s)^2}{4kt} \right) ds \]

To evaluate the integrals, we use the sifting property of the \( \delta \) function:
\[ \int_{a}^{b} \delta(s-c) g(s) = g(c) \]

for \( a < c < b \). Thus
\[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp \left( -\frac{(x+ct)^2}{4kt} \right) \]
Figure 1: Sketch of $u(x, t)$ with $c = k$, for $kt = 0.1$ (solid), 1 (dashed) and 2 (dash-dot).

Plots are given in Figure 1. The convective term $cu_x$ moves the peak to the left, as the lump becomes more spread out (diffuse) due to the diffusion term $ku_{xx}$.

(c) For the semi-infinite rod, things are different (e.g. see problem 10.5.14). First, we use the methods of PSet 2, Q5a, to transform the PDE to the basic Heat Equation,

$$u(x, t) = e^{-[x+(c/2)t]/c/2k}v(x, t)$$

so that the PDE for $u$ is transformed to

$$v_t = kv_{xx}$$  \hspace{1cm} (1)

The initial condition is

$$v(x, 0) = u(x, 0) e^{xc/2k} = f(x) e^{xc/2k}$$  \hspace{1cm} (2)

and the BC is

$$0 = \frac{\partial u}{\partial x}(0, t) = e^{-(c^2/4k)t} \left( -\frac{c}{2k}v(0, t) + \frac{\partial v}{\partial x}(0, t) \right)$$

Thus

$$0 = -\frac{c}{2k}v(0, t) + \frac{\partial v}{\partial x}(0, t)$$  \hspace{1cm} (3)
We extend \( v(x,t) \) to the infinite rod \(-\infty < x < \infty\), and let’s suppose the IC is \( v(x,0) = \tilde{f}(x) \). The solution to the PDE (1) and the IC is, from class,

\[
v(x,t) = \int_{-\infty}^{\infty} \hat{f}(s) \exp\left(-\frac{(x-s)^2}{4kt}\right) ds
\]

We now have to choose \( \tilde{f}(x) \) to satisfy the BC (3). First, compute the following:

\[
v(0,t) = \int_{-\infty}^{\infty} \frac{s}{2k} \hat{f}(s) \exp\left(-\frac{s^2}{4kt}\right) ds
\]

\[
v_x(0,t) = \int_{-\infty}^{\infty} \frac{1}{2k} \hat{f}(s) \exp\left(-\frac{s^2}{4kt}\right) ds
\]

Thus

\[
-\frac{c}{2k} v(0,t) + \frac{\partial v}{\partial x}(0,t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{2k\sqrt{4\pi kt}} \left(-c+s\right) \exp\left(-\frac{s^2}{4kt}\right) ds \quad (4)
\]

So if we define

\[
\tilde{f}(x) = \begin{cases} 
  f(x) e^{xc/2k}, & x \geq 0, \\
  -f(-x) e^{-xc/2k-cx/t+c-x/t}, & x < 0,
\end{cases}
\]

the integrand in (4) is odd, so that

\[
-\frac{c}{2k} v(0,t) + \frac{\partial v}{\partial x}(0,t) = 0.
\]

Note that \( \tilde{f}(x) \) is neither even nor odd, but by choosing it we satisfy the BC (4). Also, for \( x > 0 \), \( \tilde{f}(x) = f(x) e^{xc/2k} \), which is the IC (2) for \( v(x,t) \). Now with \( f(x) = \delta(x-1) \), we have

\[
\tilde{f}(x) = \begin{cases} 
  \delta(x-1) e^{xc/2k}, & x \geq 0, \\
  -\delta(-x-1) e^{-xc/2k-cx/t+c-x/t}, & x < 0,
\end{cases}
\]

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and hence
\[
v(x, t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{\sqrt{4\pi kt}} \exp \left( -\frac{(x - s)^2}{4kt} \right) ds
\]
\[
= -\int_{-\infty}^{0} \frac{\delta(-s-1) e^{-sc/2k} - c - s/t}{\sqrt{4\pi kt}} \exp \left( -\frac{(x - s)^2}{4kt} \right) ds
\]
\[
+ \int_{0}^{\infty} \frac{\delta(s-1) e^{sc/2k}}{\sqrt{4\pi kt}} \exp \left( -\frac{(x - s)^2}{4kt} \right) ds
\]
\[
= -\int_{-\infty}^{0} \frac{\delta(-s-1) e^{-sc/2k} - c - s/t}{\sqrt{4\pi kt}} \exp \left( -\frac{(x - s)^2}{4kt} \right) ds
\]
\[
+ \int_{0}^{\infty} \frac{\delta(s-1) e^{sc/2k}}{\sqrt{4\pi kt}} \exp \left( -\frac{(x - s)^2}{4kt} \right) ds
\]
\[
= \frac{e^{c/2k}}{\sqrt{4\pi kt}} \left( \frac{c - 1/t}{c + 1/t} \exp \left( -\frac{(x + 1)^2}{4kt} \right) + \exp \left( -\frac{(x - 1)^2}{4kt} \right) \right)
\]

Thus
\[
u(x, t) = e^{-|x+(c/2)t|c/2k} v(x, t)
\]
\[
= \frac{e^{-|x+(c/2)t|+1|c/2k}}{\sqrt{4\pi kt}} \left( \frac{c - 1/t}{c + 1/t} \exp \left( -\frac{(x + 1)^2}{4kt} \right) + \exp \left( -\frac{(x - 1)^2}{4kt} \right) \right)
\]
is the solution of the Heat Equation with Convection on the semi-infinite rod, insulated at \(x = 0\). Plots are given in Figure 2.

2 Problem 2

Do problem 10.6.4 in Haberman (p 499-500), both (a) and (b). The answer for (a) is in the back - please show how to get that answer. You may find sections 10.5 and 10.6 in Haberman useful as reference reading.

Solutions: Solve Laplace's equation on the half plane,
\[
\nabla^2 u = 0, \quad x > 0, \quad y > 0
\]
subject to the BCs
\[
u(0, y) = 0
\]
and either (a)
\[
\frac{\partial u}{\partial y} (x, 0) = f(x)
\]
or (b)

\[ u(x,0) = f(x) \]

Since \( u = 0 \) along \( y = 0 \), we must extend \( f(x) \) to be odd,

\[ \tilde{f}(x) = \begin{cases} f(x), & x \geq 0, \\ -f(-x), & x < 0. \end{cases} \]

We now solve Laplace’s equation on the half plane \( \{y \geq 0, -\infty < x < \infty\} \), as in §3 of the Notes,

\[ \nabla^2 \tilde{u} = 0, \quad -\infty < x < \infty, \quad y > 0 \]
\[ \tilde{u}(x,0) = \tilde{f}(x), \quad -\infty < x < \infty, \]
\[ \tilde{u}(0,y) = 0, \quad y > 0 \]

Since the inhomogeneous BC is imposed along the \( x \)-axis, we employ the Fourier Transform in \( x \),

\[ \mathcal{F}\left[g(x,y)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x,y) e^{i\omega x} dx \]
and define $\bar{U}(\omega, y) = \mathcal{F}[\bar{u}(x, y)]$. As before, we have

$$
\mathcal{F}[\dddot{u}_{xx}] = -\omega^2 \mathcal{F}[\dddot{u}] = -\omega^2 \bar{U}(\omega, y), \quad \mathcal{F}[^{\bar{u}_{yy}}] = \frac{\partial^2}{\partial y^2} \mathcal{F}[\dddot{u}] = \frac{\partial^2}{\partial y^2} \bar{U}(\omega, y).
$$

Hence Laplace’s equation becomes

$$
\frac{\partial^2}{\partial y^2} \bar{U}(\omega, y) - \omega^2 \bar{U}(\omega, y) = 0
$$

Solving the ODE and being careful about the fact that $\omega$ can be positive or negative, we have

$$
\bar{U}(\omega, y) = c_1(\omega) e^{-|\omega|y} + c_2(\omega) e^{|\omega|y}
$$

where $c_1(\omega)$, $c_2(\omega)$ are arbitrary functions. Since the temperature must remain bounded as $y \to \infty$, we must have $c_2(\omega) = 0$. Thus

$$
\bar{U}(\omega, y) = c_1(\omega) e^{-|\omega|y} \tag{5}
$$

(a) Imposing the BC at $y = 0$ gives

$$
-|\omega| c_1(\omega) = \left. \frac{\partial}{\partial y} \bar{U}(\omega, y) \right|_{y=0} = \mathcal{F} \left[ \frac{\partial}{\partial y} \dddot{u}(x, 0) \right] = \mathcal{F}[f(x)]
$$

Thus

$$
\bar{U}(\omega, y) = \mathcal{F}[f(x)] \frac{e^{-|\omega|y}}{-|\omega|}
$$

Note that the IFT of $e^{-|\omega|y} / (-|\omega|)$ is

$$
\mathcal{F}^{-1} \left[ \frac{e^{-|\omega|y}}{-|\omega|} \right] = \int_{-\infty}^{\infty} e^{-|\omega|y} e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-|\omega|y} dy \right) e^{-i\omega x} d\omega = \int \left( \int_{-\infty}^{\infty} e^{-|\omega|y} e^{-i\omega x} d\omega \right) dy = \int \mathcal{F}^{-1} [e^{-|\omega|y}] dy
$$

In the text and in section 3 of the notes, we showed that

$$
\mathcal{F}^{-1} [e^{-|\omega|y}] = \frac{2y}{x^2 + y^2}
$$

Thus

$$
\mathcal{F}^{-1} \left[ \frac{e^{-|\omega|y}}{-|\omega|} \right] = \int \left( \frac{2y}{x^2 + y^2} \right) dy = \ln \left( x^2 + y^2 \right)
$$
Therefore, applying the Convolution Theorem with \( \mathcal{F}^{-1} [c_1(\omega)] = \tilde{f}(x) \) and \( \mathcal{F}^{-1} [e^{-|\omega|y}/(-|\omega|)] \) gives

\[
\tilde{u}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(s) \ln ((x-s)^2 + y^2) \, ds \\
= \frac{1}{2\pi} \int_{-\infty}^{0} \tilde{f}(s) \ln ((x-s)^2 + y^2) \, ds + \frac{1}{2\pi} \int_{0}^{\infty} \tilde{f}(s) \ln ((x-s)^2 + y^2) \, ds \\
= -\frac{1}{2\pi} \int_{-\infty}^{0} f(-s) \ln ((x-s)^2 + y^2) \, ds + \frac{1}{2\pi} \int_{0}^{\infty} f(s) \ln ((x-s)^2 + y^2) \, ds \\
= \frac{1}{2\pi} \int_{-\infty}^{0} f(s) \ln ((x+s)^2 + y^2) \, ds + \frac{1}{2\pi} \int_{0}^{\infty} f(s) \ln ((x-s)^2 + y^2) \, ds \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \ln \left( \frac{(x-s)^2 + y^2}{(x+s)^2 + y^2} \right) \, ds
\]

(b) Imposing the BC at \( y = 0 \) gives

\[
c_1(\omega) = \tilde{U}(\omega, 0) = \mathcal{F} [\tilde{u}(x, 0)] = \mathcal{F} [\tilde{f}(x)].
\]

Therefore, applying the Convolution Theorem with \( \mathcal{F}^{-1} [c_1(\omega)] = \tilde{f}(x) \) and \( \mathcal{F}^{-1} [e^{-|\omega|y}] = 2y/(x^2 + y^2) \) gives

\[
\tilde{u}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(s) \frac{2y}{(x-s)^2 + y^2} \, ds
\]

In both (a) and (b), limiting \( x \geq 0 \) gives the solution to Laplace’s equation on the quarter plane,

\[
u(x, y) = \tilde{u}(x, y), \quad x \geq 0.
\]

You don’t have to, but you can rearrange this some more,

\[
u(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{0} f(-s) \frac{2y}{(x-s)^2 + y^2} \, ds + \frac{1}{2\pi} \int_{0}^{\infty} f(s) \frac{2y}{(x-s)^2 + y^2} \, ds \\
= \frac{1}{2\pi} \int_{-\infty}^{0} f(s) \frac{2y}{(x+s)^2 + y^2} \, ds + \frac{1}{2\pi} \int_{0}^{\infty} f(s) \frac{2y}{(x-s)^2 + y^2} \, ds \\
= \frac{y}{\pi} \int_{0}^{\infty} f(s) \left( \frac{-1}{(x+s)^2 + y^2} + \frac{1}{(x-s)^2 + y^2} \right) \, ds \\
= \frac{4xy}{\pi} \int_{0}^{\infty} s f(s) \, ds \\
\frac{2y}{(x+s)^2 + y^2} \left( \frac{1}{(x-s)^2 + y^2} \right)
\]