Solution to Problems for the 1-D Wave Equation
18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Suppose that an “infinite string” has an initial displacement
\[
  u(x, 0) = f(x) = \begin{cases} 
  x + 1, & -1 \leq x \leq 0 \\
  1 - 2x, & 0 \leq x \leq 1/2 \\
  0, & x < -1 \text{ and } x > 1/2
  \end{cases}
\]
and zero initial velocity \(u_t(x, 0) = 0\). Write down the solution of the wave equation
\[
  u_{tt} = u_{xx}
\]
with ICs \(u(x, 0) = f(x)\) and \(u_t(x, 0) = 0\) using D’Alembert’s formula. Illustrate
the nature of the solution by sketching the \(ux\)-profiles \(y = u(x, t)\) of the string
displacement for \(t = 0, 1/2, 1, 3/2\).

**Solution:** D’Alembert’s formula is
\[
  u(x, t) = \frac{1}{2} \left( f(x - t) + f(x + t) + \int_{x-t}^{x+t} g(s) \, ds \right)
\]
In this case \(g(s) = 0\) so that
\[
  u(x, t) = \frac{1}{2} (f(x - t) + f(x + t))
\]
(1)
The problem reduces to adding shifted copies of \(f(x)\) and then plotting the associated
\(u(x, t)\). To determine where the functions overlap or where \(u(x, t)\) is zero, we plot
the characteristics \(x \pm t = -1\) and \(x \pm t = 1/2\) in the space time plane \((xt)\) in Figure
1.

For \(t = 0\), (1) becomes
\[
  u(x, 0) = \frac{1}{2} (f(x) + f(x)) = f(x)
\]
For $t = 1/2$, (1) becomes

$$u(x,t) = \frac{1}{2} \left( f\left(x - \frac{1}{2}\right) + f\left(x + \frac{1}{2}\right) \right)$$

Note that

$$f\left(x - \frac{1}{2}\right) = \begin{cases} (x - \frac{1}{2}) + 1, & -1 \leq (x - \frac{1}{2}) \leq 0 \\ 1 - 2 \left(x - \frac{1}{2}\right), & 0 \leq (x - \frac{1}{2}) \leq 1/2 \\ 0, & (x - \frac{1}{2}) < -1 \text{ and } (x - \frac{1}{2}) > 1/2 \end{cases}$$

and similarly,

$$f\left(x + \frac{1}{2}\right) = \begin{cases} x + \frac{3}{2}, & -\frac{3}{2} \leq x \leq -\frac{1}{2} \\ -2x, & -\frac{1}{2} \leq x \leq 0 \\ 0, & x < -\frac{3}{2} \text{ and } x > 0 \end{cases}$$

Thus, over the region $-\frac{1}{2} \leq x \leq 0$ we have to be careful about adding the two
functions; in the other regions either one or both functions are zero. We have
\[ u(x, \frac{1}{2}) = \frac{1}{2} \left( f \left( x - \frac{1}{2} \right) + f \left( x + \frac{1}{2} \right) \right) \]
\[ = \begin{cases} 
\frac{x}{2} + \frac{3}{4}, & -\frac{3}{2} \leq x \leq -\frac{1}{2} \\
-\frac{x}{2} + \frac{1}{4}, & -\frac{1}{2} \leq x \leq 0 \\
\frac{x}{2} + \frac{1}{4}, & 0 \leq x \leq \frac{1}{2} \\
x - 1, & \frac{1}{2} \leq x \leq 1 \\
0, & x < -\frac{3}{2} \text{ and } x > 1 
\end{cases} \]

For \( t = 1 \), your plot of the characteristics shows that \( f(x - 1) \) and \( f(x + 1) \) do not overlap, so you just have to worry about the different regions. Note that
\[ f(x + 1) = \begin{cases} 
(x + 1) + 1, & -1 \leq x + 1 \leq 0 \\
1 - 2(x + 1), & 0 \leq x + 1 \leq 1/2 \\
0, & x + 1 < -1 \text{ and } x + 1 > 1/2 
\end{cases} \]
\[ = \begin{cases} 
x + 2, & -2 \leq x \leq -1 \\
-1 - 2x, & -1 \leq x \leq -1/2 \\
0, & x < -2 \text{ and } x > -1/2 
\end{cases} \]
\[ f(x - 1) = \begin{cases} 
x, & 0 \leq x \leq 1 \\
3 - 2x, & 1 \leq x \leq 3/2 \\
0, & x < 0 \text{ and } x > 3/2 
\end{cases} \]
so that
\[ u(x, 1) = \frac{1}{2} (f(x - 1) + f(x + 1)) \]
\[ = \begin{cases} 
\frac{x}{2} + 1, & -2 \leq x \leq -1 \\
-\frac{1}{2} - x, & -1 \leq x \leq -1/2 \\
\frac{x}{2}, & 0 \leq x \leq 1 \\
\frac{3}{2} - x, & 1 \leq x \leq 3/2 \\
0, & x < -2, -1/2 < x < 0, \text{ and } x > 3/2 
\end{cases} \]

For \( t = 3/2 \), the forward and backward waves are even further apart, and
\[ f \left( x - \frac{3}{2} \right) = \begin{cases} 
x - \frac{1}{2}, & \frac{1}{2} \leq x \leq \frac{3}{2} \\
4 - 2x, & \frac{3}{2} \leq x \leq 2 \\
0, & x < \frac{1}{2} \text{ and } x > 2 
\end{cases} \]
\[ f \left( x + \frac{3}{2} \right) = \begin{cases} 
x + \frac{5}{2}, & -\frac{5}{2} \leq x \leq -\frac{3}{2} \\
-2 - 2x, & -\frac{3}{2} \leq x \leq -1 \\
0, & x < -\frac{5}{2} \text{ and } x > -1 
\end{cases} \]
and hence
\[
\begin{align*}
  u \left( x, \frac{3}{2} \right) &= \frac{1}{2} \left( f \left( x - \frac{3}{2} \right) + f \left( x + \frac{3}{2} \right) \right) \\
  &= \begin{cases} 
    x + \frac{5}{4}, & -\frac{5}{2} \leq x \leq -\frac{3}{2}, \\
    -1 - x, & -\frac{3}{2} < x \leq -1, \\
    2 - x, & \frac{1}{2} \leq x \leq \frac{3}{2}, \\
    0, & x < -\frac{5}{2}, \quad -1 < x < \frac{3}{2}, \text{ and } x > 2
  \end{cases}
\end{align*}
\]

The solution \( u(x, t_0) \) is plotted at times \( t_0 = 0, 1/2, 1, 3/2 \) in Figure 2. A 3D version of \( u(x, t) \) is plotted in Figure 3.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

\[
  u_t (x, 0) = g (x) = \begin{cases} 
    -1, & -1 \leq x < 0 \\
    1, & 0 \leq x \leq 1 \\
    0, & x < -1 \text{ and } x > 1
  \end{cases}
\]

Solution: D’Alembert’s formula is

\[
  u (x, t) = \frac{1}{2} \left( f (x - t) + f (x + t) + \int_{x-t}^{x+t} g (s) \, ds \right)
\]
In this case $f(s) = 0$ so that

$$u(x, t) = \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds$$

The problem reduces to noting where $x \pm t$ lie in relation to $\pm 1$ and evaluating the integral. These characteristics are plotted in Figure 1 in the notes.

You can proceed in two ways. First, you can draw two more characteristics $x \pm t = 0$ so you can decide where the integration variable $s$ is with respect to zero, and hence if $g(s) = -1$ or 1. The second way is to note that for $a < b$ and $|a|, |b| < 1$,

$$\int_{a}^{b} g(s) \, ds = |b| - |a|$$

for positive and negative $a, b$. I’ll use the second method; the answers you get from the first are the same.

In Region $R_1$,

$$|x \pm t| \leq 1$$
and hence there are 3 cases: $x - t < 0$, $x$

\[ u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds \]

\[ = \frac{1}{2} (|x + t| - |x - t|) \]

In Region $R_2$, $x + t > 1$ and $-1 < x - t < 1$, so that

\[ u(x,t) = \frac{1}{2} \left( \int_{x-t}^{x-t} + \int_{x-t}^{x+t} \right) g(s) \, ds = \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds \]

\[ = \frac{1}{2} (1 - |x - t|) \]

In Region $R_3$, $x - t < -1$ and $-1 < x + t < 1$, so that

\[ u(x,t) = \frac{1}{2} \left( \int_{x-t}^{x-t} + \int_{x-t}^{x+t} \right) g(s) \, ds = \frac{1}{2} \int_{x-t}^{x-t} g(s) \, ds = \frac{1}{2} (|x + t| - |-1|) \]

\[ = \frac{1}{2} (|x + t| - 1) \]

In Region $R_4$, $x + t > 1$ and $x - t < -1$, so that

\[ u(x,t) = \frac{1}{2} \left( \int_{x-t}^{x-t} + \int_{x-t}^{x+t} \right) g(s) \, ds \]

\[ = \frac{1}{2} \int_{x-t}^{x-t} g(s) \, ds = \frac{1}{2} (-1 + 1) \]

\[ = 0 \]

In Region $R_5$, $x + t < -1$ and hence $u(x,t) = 0$. In region $R_6$, $x - t > 1$, so that

\[ u(x,t) = 0 \]

At $t = 0$,

\[ u(x,0) = \frac{1}{2} \int_{x}^{x} g(s) \, ds = 0 \]

At $t = 1/2$, the regions $R_n$ are given in the notes and

\[ u \left( x, \frac{1}{2} \right) = \begin{cases} \frac{1}{2} (|x + \frac{1}{2}| - |x - \frac{1}{2}|), & x \in R_1 = \left[ -\frac{1}{2}, \frac{1}{2} \right] \\ \frac{1}{2} \left( 1 - |x - \frac{1}{2}| \right), & x \in R_2 = \left[ \frac{1}{2}, \frac{3}{2} \right] \\ \frac{1}{2} (|x + \frac{1}{2}| - 1), & x \in R_3 = \left[ \frac{3}{2}, \frac{5}{2} \right] \\ 0, & x \in R_5, R_6 = \{|x| > 3/2\} \end{cases} \]

The absolute values are easy to resolve (i.e. write without them) in this case. For example, for $x \in [-1/2, 1/2]$, we have $|x - 1/2| = -(x - 1/2)$. Thus,

\[ u \left( x, \frac{1}{2} \right) = \begin{cases} x, & x \in R_1 = \left[ -\frac{1}{2}, \frac{1}{2} \right] \\ \frac{3}{4} - \frac{x}{2}, & x \in R_2 = \left[ \frac{1}{2}, \frac{3}{2} \right] \\ -\frac{3}{4} - \frac{x}{2}, & x \in R_3 = \left[ \frac{3}{2}, \frac{5}{2} \right] \\ 0, & x \in R_5, R_6 = \{|x| > 3/2\} \]
At $t = 1$, the regions $R_n$ are given in the notes and 

$$u(x, 1) = \begin{cases} 
\frac{1}{2}(1 - |x - 1|), & x \in R_2 = [0, 2], \\
\frac{1}{2}(|x + 1| - 1), & x \in R_3 = [-2, 0], \\
0, & x \in R_5, R_6 = \{|x| > 3/2\}.
\end{cases}$$

You could leave your answer like this, or write it without absolute values (have to divide $[0, 2]$ and $[-2, 0]$ into cases):

$$u(x, 1) = \begin{cases} 
x/2, & x \in [0, 1], \\
\frac{1}{2}(2 - x), & x \in [1, 2], \\
-\frac{1}{2}(x + 2), & x = [-2, -1] \\
x/2, & x = [-1, 0], \\
0, & x \in R_5, R_6 = \{|x| > 3/2\}.
\end{cases}$$

At $t = 3/2$, the regions $R_n$ are not given explicitly, but can be found from Figure 1 in the notes by noting where the line $t = 3/2$ crosses each region:

$$u\left(x, \frac{3}{2}\right) = \begin{cases} 
\frac{1}{2}(1 - |x - \frac{3}{2}|), & x \in R_2 = \left[\frac{1}{2}, \frac{5}{2}\right], \\
\frac{1}{2}(|x + \frac{3}{2}| - 1), & x \in R_3 = \left[-\frac{5}{2}, -\frac{1}{2}\right], \\
0, & x \in R_4, R_5, R_6 = \{|x| > 5/2 \text{ or } |x| < 1/2\}
\end{cases}$$

Again, you could leave your answer like this, or write it without absolute values (have to divide $[1/2, 5/2]$ and $[-5/2, -1/2]$ into cases):

$$u\left(x, \frac{3}{2}\right) = \begin{cases} 
\frac{1}{2}\left(x - \frac{1}{2}\right), & x \in R_2 = \left[\frac{1}{2}, \frac{3}{2}\right], \\
\frac{1}{2}\left(x + \frac{1}{2}\right), & x \in R_2 = \left[\frac{3}{2}, \frac{5}{2}\right], \\
-\frac{1}{2}(x + \frac{1}{2}), & x \in R_3 = \left[-\frac{5}{2}, -\frac{3}{2}\right], \\
\frac{1}{2}(x + \frac{1}{2}), & x \in R_3 = \left[-\frac{3}{2}, -\frac{1}{2}\right], \\
0, & x \in R_4, R_5, R_6 = \{|x| > 5/2 \text{ or } |x| < 1/2\}
\end{cases}$$

The solution $u(x, t_0)$ is plotted at times $t_0 = 0, 1/2, 1, 3/2$ in Figure 4.

## 2 Problem 2

(i) For an infinite string (i.e. we don’t worry about boundary conditions), what initial conditions would give rise to a purely forward wave? Express your answer in terms of the initial displacement $u(x, 0) = f(x)$ and initial velocity $u_t(x, 0) = g(x)$ and their derivatives $f'(x), g'(x)$. Interpret the result intuitively.

**Solution:** Recall in class that we write D’Alembert’s solution as

$$u(x, t) = P(x - t) + Q(x + t)$$  \hspace{1cm} (2)
Figure 4: Plot of $u(x, t_0)$ for $t_0 = 0, 1/2, 1, 3/2$ for 1(b).

where

$$Q(x) = \frac{1}{2} \left( f(x) + \int_0^x g(s) \, ds + Q(0) - P(0) \right) \quad (3)$$

$$P(x) = \frac{1}{2} \left( f(x) - \int_0^x g(s) \, ds - Q(0) + P(0) \right) \quad (4)$$

To only have a forward wave, we must have

$$Q(x) = \text{const} = q_1$$

Substituting (3) gives

$$Q(x) = q_1 = \frac{1}{2} \left( f(x) + \int_0^x g(s) \, ds + Q(0) - P(0) \right)$$

Differentiating in $x$ gives

$$0 = \frac{1}{2} \left( \frac{df}{dx} + g(x) \right)$$

Thus

$$g(x) = -\frac{df}{dx} \quad (5)$$
Substituting (5) into (3) gives

\[ Q(x) = \frac{1}{2} \left( f(0) + Q(0) - P(0) \right) \]

and setting \( x = 0 \) yields \( f(0) - P(0) = Q(0) \). Substituting this and (5) into (4) gives

\[ P(x) = \frac{1}{2} (2f(x) - f(0) - Q(0) + P(0)) = f(x) \]

and hence

\[ u(x,t) = f(x-t). \]

The displacement \( u(x,t) \) only contains the forward wave! Intuitively, we have set the initial velocity of the string in such a way, given by Eq. (5), as to cancel the backward wave.

(ii) Again for an infinite string, suppose that \( u(x,0) = f(x) \) and \( u_t(x,0) = g(x) \) are zero for \(|x| > a\), for some real number \( a > 0 \). Prove that if \( t+x > a \) and \( t-x > a \), then the displacement \( u(x,t) \) of the string is constant. Relate this constant to \( g(x) \).

Solution: D’Alembert’s solution for the wave equation is

\[ u(x,t) = \frac{1}{2} \left( f(x - t) + f(x + t) \right) + \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds \]

If \( x + t > a \) and \( t - x > a \) (this is the Region \( R_4! \)), then \(|x+t| > a\) and \(|x-t| > a\), so that \( f(x \pm t) = 0 \). Furthermore, with \( x - t < -a \) and \( x + t > a \) we have

\[ \int_{x-t}^{x+t} g(s) \, ds = \int_{-a}^{a} g(s) \, ds = \int_{-\infty}^{\infty} g(s) \, ds = c_a \]

Thus \( c_a \) is just the area under the curve \( g(x) \), and

\[ u(x,t) = \frac{c_a}{2}, \quad x + t > a, \quad t - x > a. \]

3 Problem 3

Consider a semi-infinite vibrating string. The vertical displacement \( u(x,t) \) satisfies

\[ \begin{align*}
& u_{tt} = u_{xx}, \quad x \geq 0, \quad t \geq 0 \\
& u(0,t) = 0, \quad t \geq 0 \\
& u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad x \geq 0,
\end{align*} \]

The BC at infinity is that \( u(x,t) \) must remain bounded as \( x \to \infty \).
(a) Show that D’Alembert’s formula solves (6) when \( f(x) \) and \( g(x) \) are extended to be odd functions.

**Solution:** Let \( \hat{f}(x) \) and \( \hat{g}(x) \) be the odd extensions of \( f(x) \) and \( g(x) \), respectively,

\[
\hat{f}(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases}, \quad \hat{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0 \end{cases}
\]

You can check for yourself that \( \hat{f}(x) \) and \( \hat{g}(x) \) are odd functions, i.e. \( \hat{f}(-x) = -\hat{f}(x) \) and \( \hat{g}(-x) = -\hat{g}(x) \). We now write D’Alembert’s solution with \( \hat{f}(x) \) and \( \hat{g}(x) \) replacing \( f(x) \) and \( g(x) \):

\[
u(x, t) = \frac{1}{2} \left( \hat{f}(x-t) + \hat{f}(x+t) + \int_{x-t}^{x+t} \hat{g}(s) \, ds \right) \tag{7}
\]

Eq. (7) is D’Alembert’s solution for the following wave problem on the infinite string:

\[
u_{tt} = \nu_{xx}, \quad -\infty < x < \infty, \quad t \geq 0
\]

\[
u(x, 0) = \hat{f}(x), \quad \frac{\partial \nu}{\partial t}(x, 0) = \hat{g}(x), \quad -\infty < x < \infty.
\]

Hence we know (7) satisfies the wave equation, by the way we found D’Alembert’s formula. Of course, you can check that directly:

\[
u_x = \frac{1}{2} \left( \hat{f}'(x-t) + \hat{f}'(x+t) + \hat{g}'(x+t) - \hat{g}'(x-t) \right)
\]

\[
u_{xx} = \frac{1}{2} \left( \hat{f}''(x-t) + \hat{f}''(x+t) + \hat{g}'(x+t) - \hat{g}'(x-t) \right)
\]

\[
u_t = \frac{1}{2} \left( \hat{f}'(x-t)(-1) + \hat{f}'(x+t) + \hat{g}(x+t) - \hat{g}(x-t)(-1) \right)
\]

\[
u_{tt} = \frac{1}{2} \left( \hat{f}''(x-t)(-1)^2 + \hat{f}''(x+t) + \hat{g}'(x+t) - \hat{g}'(x-t)(-1)^2 \right)
\]

Thus \( \nu_{tt} = \nu_{xx} \). Also, for \( x \geq 0 \),

\[
u(x, 0) = \hat{f}(x) = f(x)
\]

\[
u_t(x, 0) = \hat{g}(x) = g(x)
\]

Thus (7) satisfies the ICs. Lastly,

\[
u(0, t) = \frac{1}{2} \left( \hat{f}(-t) + \hat{f}(t) + \int_{-t}^{t} \hat{g}(s) \, ds \right)
\]

But since \( \hat{f} \) is odd, \( \hat{f}(-t) = -\hat{f}(t) \) and since \( \hat{g}(s) \) is odd, the integral of \( \hat{g}(s) \) over a region symmetric about the origin is zero! Hence

\[
u(0, t) = \frac{1}{2} \left( -\hat{f}(t) + \hat{f}(t) + 0 \right) = 0
\]
which verifies (7) satisfies the fixed string \((u = 0)\) BC at \(x = 0\).

(b) Let

\[
f(x) = \begin{cases} 
\sin^2(\pi x), & 1 \leq x \leq 2 \\
0, & 0 \leq x \leq 1, \quad x \geq 2
\end{cases}
\]

and \(g(x) = 0\) for \(x \geq 0\). Sketch \(u\) vs. \(x\) for \(t = 0, 1, 2, 3\).

**Solution:** D’Alembert’s solution reduces to

\[
u(x,t) = \frac{1}{2} \left( \hat{f}(x-t) + \hat{f}(x+t) \right)
\]

Solving this reduces to finding where \(x - t\) and \(x + t\) are and whether they are negative. The important characteristics are \(x \pm t = \pm 1, \pm 2\). A drawing is useful. The characteristics are plotted in Figure 5 and the solution \(u(x, t_0)\) at times \(t_0 = 0, 1, 2, 3\) in Figure 6.
4 Problem 4

The acoustic pressure in an organ pipe obeys the 1-D wave equation (in physical variables)

\[ p_{tt} = c^2 p_{xx} \]

where \( c \) is the speed of sound in air. Each organ pipe is closed at one end and open at the other. At the closed end, the BC is that \( p_x(0, t) = 0 \), while at the open end, the BC is \( p(l, t) = 0 \), where \( l \) is the length of the pipe.

(a) Use separation of variables to find the normal modes \( p_n(x, t) \).

(b) Give the frequencies of the normal modes and sketch the pressure distribution for the first two modes.

(c) Given initial conditions \( p(x, 0) = f(x) \) and \( p_t(x, 0) = g(x) \), write down the general initial boundary value problem (PDE, BCs, ICs) for the organ pipe and determine the series solutions.

**Solution:** Separate variables

\[ p_n(x, t) = X(x) T(t) \]
so that the PDE becomes
\[ \frac{T''}{c^2 T} = \frac{X''}{X} \]
and since the left side is a function of \( t \) only and the right a function of \( x \) only, then both sides equal a constant \(-\lambda\):
\[ \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \]
The boundary conditions are
\[ 0 = \frac{\partial p}{\partial x} (0, t) = X'(0) T(t), \quad 0 = p(l, t) = X(l) T(t) \]
For a non-trivial solution, we must have \( X'(0) = 0 \) and \( X(l) = 0 \). We obtain the Sturm Liouville problem
\[ X'' + \lambda X = 0; \quad X'(0) = 0 = X(l) \]
By replacing \( x \) with \( x/l \) in problem 4 on assignment 1, the eigenfunctions and eigenvalues are
\[ X_n(x) = \cos \left( \frac{2n-1}{2} \frac{\pi}{l} x \right), \quad \lambda_n = \frac{(2n-1)^2 \pi^2}{4l^2}, \quad n = 1, 2, 3, ... \]
The corresponding time functions are
\[ T_n(t) = \alpha_n \cos \left(c\sqrt{\lambda_n} t \right) + \beta_n \sin \left(c\sqrt{\lambda_n} t \right) \]
Thus the normal modes are
\[ p_n(x, t) = X_n(x) T_n(t) \]
\[ = \cos \left( \frac{2n-1}{2} \frac{\pi}{l} x \right) \left( \alpha_n \cos \left( \frac{2n-1}{2l} \pi c t \right) + \beta_n \sin \left( \frac{2n-1}{2l} \pi c t \right) \right) \]
\[ = \gamma_n \cos \left( \frac{2n-1}{2} \frac{\pi}{l} x \right) \cos \left( \frac{2n-1}{2l} \pi c t - \psi_n \right) \]
where \( \gamma_n = \sqrt{\alpha_n^2 + \beta_n^2} \) and \( \psi_n = \arctan(\beta_n/\alpha_n) \).
(b) The angular frequency \( \omega_n \) of the \( n \)'th mode is
\[ \omega_n = \frac{2n-1}{2l} \pi c \]
and thus the frequency of the \( n \)'th mode is
\[ f_n = \frac{\omega_n}{2\pi} = \frac{2n-1}{4} \frac{c}{l} \]
Thus, the frequencies and pressure distribution for the first two normal modes \( n = 1, 2 \) are

\[
f_1 = \frac{1}{4l} c, \quad p_1(x, t) = \gamma_1 \cos\left(\frac{\pi x}{2l}\right) \cos\left(\frac{\pi ct}{2l} - \psi_1\right)
\]

\[
f_2 = \frac{3}{4l} = 3f_1, \quad p_2(x, t) = \gamma_2 \cos\left(\frac{3\pi x}{2l}\right) \cos\left(\frac{3\pi ct}{2l} - \psi_n\right)
\]

Various phases of the pressure distributions \( p_n(x, t) \) of the first two normal modes are plotted in Figure 7, with \( \gamma_n = 1 \). Notice that \( \partial p/\partial x = 0 \) at the close end \( (x = 0) \) and \( p = 0 \) at the right end \( (x = l) \). This are like the standing waves that appear when you shake a rope at \( x = 0 \) attached to a wall at \( x = l \).

(c) The general initial boundary value problem for the organ pipe is

\[
p_{tt} = c^2 p_{xx}, \quad 0 < x < l, \quad t > 0
\]

\[
\frac{\partial p}{\partial x}(0, t) = 0 = p(l, t), \quad t > 0,
\]

\[
p(x, 0) = f(x), \quad \frac{\partial p}{\partial t}(x, 0) = g(x), \quad 0 < x < l.
\]

Continuing from above, we including all the modes \( p_n(x, t) \) in our series solution for
\[ p(x,t), \]

\[ p(x,t) = \sum_{n=1}^{\infty} p_n(x,t) = \sum_{n=1}^{\infty} \cos \left( \frac{2n-1}{2} \frac{x}{l} \right) \left( \alpha_n \cos \left( \frac{2n-1}{2l} \pi ct \right) + \beta_n \sin \left( \frac{2n-1}{2l} \pi ct \right) \right) \]

Imposing the ICs gives

\[ f(x) = p(x,0) = \sum_{n=1}^{\infty} \cos \left( \frac{2n-1}{2} \frac{x}{l} \right) \alpha_n \]

\[ g(x) = \frac{\partial p}{\partial t}(x,0) = \sum_{n=1}^{\infty} \cos \left( \frac{2n-1}{2} \frac{x}{l} \right) \frac{2n-1}{2l} c \pi \beta_n \]

These are both cosine series. Multiplying each side by \( \cos \left( \frac{(2m-1) \pi x}{2l} \right) \) and integrating from \( x = 0 \) to \( x = l \) and using orthogonality gives

\[ \alpha_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \left( \frac{2n-1}{2} \frac{x}{l} \right) dx, \]

\[ \frac{2n-1}{2l} c \pi \beta_n = \frac{2}{l} \int_{0}^{l} g(x) \cos \left( \frac{2n-1}{2} \frac{x}{l} \right) dx. \]

Thus

\[ \beta_n = \frac{4}{(2n-1) \pi c} \int_{0}^{l} g(x) \cos \left( \frac{2n-1}{2} \frac{x}{l} \right) dx. \]