In today’s class, we will:

- Analyze von Neumann’s algorithm in terms of the condition number [this work is due to Dantzig and Eppelman-Freund]
- Do a smoothed analysis of the condition number

1 von Neumann’s Linear Programming Algorithm

We consider the following formulation of linear programming:

Given $c, a_1, a_2, \ldots, a_n \in \mathbb{R}^d$

Is $c \in \text{hull}\{a_1, a_2, \ldots, a_n\}$?

In other words, does there exist $x_1, x_2, \ldots, x_n$ such that:

\[
\sum a_i x_i = c \\
\sum x_i = 1 \\
x_i \geq 0
\]

We would like to find either an $x$ s.t. $\|c - \sum a_i x_i\| < \epsilon$ if the linear program is feasible or find a hyperplane separating $c$ from $a_1, a_2, \ldots, a_n$ if it is not.

For technical reasons, we’ll assume that the origin lies within the hull.

Recall von Neumann’s algorithm for this program.

1. Let $y = Ax$
2. Choose $i$ maximizing $\langle c - y, a_i - y \rangle$
3. If $\langle c - y, a_i - y \rangle / \|c - y\| < \|c - y\|$ then return infeasible
4. Otherwise set $x'$ s.t. $y' = Ax'$ is the point on the line $\overline{ya_i}$ closest to $c$. Repeat.
The correctness of the algorithm derives from the following: 

Claim: If \( y, c \in \text{hull}\{a_1, a_2, \ldots, a_n\} \) then \( \exists i \) s.t.

\[
\langle \frac{c - y}{\|c - y\|}, a_i - y \rangle \geq \|c - y\| 
\]

2 Analysis of the algorithm

Let \( r = \text{dist}(c, \text{bdry}(\text{ch}\{a_1, a_2, \ldots, a_n\})) \)

Let \( R = \max_i \|a_i\| \)

Theorem 1 (Eppelman-Freund)

(I) If the linear program is feasible, the algorithm finds \( x \) s.t. \( \|Ax - c\| \leq \varepsilon \) in at most \( 8(R/r)^2\ln(2R/\varepsilon) \) iterations.

(II) If infeasible, the algorithm discovers this in at most \( 4(R/r)^2 \) iterations.

Proof

Part (I)

Claim: If the linear program is feasible then

\[
\|Ax' - c\| \leq \|Ax - c\| \sqrt{1 - \left(\frac{r}{2R}\right)^2}
\]

Before proving this claim, let’s see why it implies (I). Observe that

\[
\sqrt{1 - \left(\frac{r}{2R}\right)^2} \leq \sqrt{e^{-r^2/(2R)^2}}
\]

Initially,

\[
\|Ax_0 - c\| \leq 2R
\]

Therefore after \( k \) iterations,

\[
\|Ax_k - c\| \leq 2R \sqrt{e^{-k(r/2R)^2}}
\]

If we let \( 2R \sqrt{e^{-k(r/2R)^2}} < \varepsilon \) we get (I).

Proof of claim

From Fig 1, it is clear that,

\[
\text{dist}(c, y') \leq \text{dist}(c, y^\perp)
\]
By similarity of triangles\[\frac{\text{dist}(c, y)}{\text{dist}(c, y')} \leq \frac{H_1}{h}\]

Therefore\[\frac{\text{dist}(c, y')}{\text{dist}(c, y)} \leq \frac{H_1}{h} \leq \frac{\sqrt{l^2 - h'^2}}{1 - (h/l)^2}\]

Since, \(h \geq r\) and \(1 \leq R\), it follows that\[\frac{\text{dist}(c, y')}{\text{dist}(c, y)} \leq \sqrt{1 - \left(\frac{r}{R}\right)^2}\]
Part (II)

Claim (Dantzig) After $k$ iterations,

$$\|Ax - c\| \leq \frac{2R}{\sqrt{k}}$$

Again it’s not hard to show why this implies (II). Since point $y$ cannot go outside the convex hull, the algorithm will stop once

$$\frac{2R}{\sqrt{k}} \leq r$$

So the algorithm must terminate after $(2R/r)^2$ iterations.

Proof of claim

Let $h_k$ be the height (i.e., distance between $c$ and $y$) after $k^{th}$ iteration. As in the previous case, we can show that,

$$h_{k+1} \leq h_k \sqrt{1 - (h_k/2R)^2}$$

We claim that

$$h_k \leq 2R/\sqrt{k}$$

Proof by induction:

Note that $\|c\| \leq R$ since otherwise it would be trivial to decide infeasibility.

Therefore, $h_1 \leq 2R$

Let $h_k \leq 2R/\sqrt{k}$

Then

$$h_{k+1} \leq h_k \sqrt{1 - (h_k/2R)^2}$$

$$= \frac{2R}{\sqrt{k}} \sqrt{1 - 1/k}$$

$$= \frac{2R}{\sqrt{k}} \sqrt{\frac{k - 1}{k^2}}$$

$$\leq 2R/\sqrt{k + 1}$$

3 Condition number of a linear program

Given a linear program, $c, a_1, a_2, \ldots, a_n$, we define it’s condition number, $K$, as follows:
If \( c \in \text{hull}\{a_1, a_2, \ldots, a_n\} \),
\[
K(c, a_1, a_2, \ldots, a_n) = \inf \{ \| \Delta c \| + \sum_i a_i : c + \Delta c \notin \text{hull}\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\} \}
\]
In other words, \( K \) measures the smallest change to infeasibility. Similarly,
If \( c \notin \text{hull}\{a_1, a_2, \ldots, a_n\} \),
\[
K(c, a_1, a_2, \ldots, a_n) = \inf \{ \| \Delta c \| + \sum_i a_i : c + \Delta c \in \text{hull}\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\} \}
\]

**Claim 2** \( K(c, a_1, a_2, \ldots, a_n) = r \)

It is obvious that \( K(c, a_1, a_2, \ldots, a_n) \leq r \)
We'll try to prove \( K(c, a_1, a_2, \ldots, a_n) \geq r \)

**Lemma 3** Let \( C = \text{hull}\{a_1, a_2, \ldots, a_n\}, C' = \text{hull}\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\} \)

1. \( \gamma(C, C') \leq \max_i \| \Delta a_i \| \)
2. \( \gamma(\tilde{C}, \tilde{C}') \leq \max_i \| \Delta a_i \| \)
3. \( \gamma(\text{bdry}(C), \text{bdry}(C')) \leq \max_i \| \Delta a_i \| \)

where \( \gamma(A, B) = \max_{x \in A} \min_{y \in B} \text{dist}(x, y) \)

**Proof**

1. Let \( x \in C \)
   \[ \Rightarrow \exists \alpha_1, \alpha_2, \ldots, \alpha_n \text{ s.t.} \]
   \[
   \sum_i \alpha_i = 1 \\
   \alpha_i \geq 0 \\
   x = \sum_i \alpha_i a_i
   \]

   Let \( x' = \sum \alpha_i (a_i + \Delta a_i) \in C' \). Then,
   \[
   \text{dist}(x, x') = \| \sum \alpha_i \Delta a_i \| \\
   \leq (\sum \alpha_i) \max_i \| \Delta a_i \| \\
   = \max_i \| \Delta a_i \|
   \]
2. This is similar to 1.

3. Let $x \in \text{bdry}(C)$
   \[
   \exists x_1 \in C' \text{ s.t. } \text{dist}(x, x_1) \leq \max_i \|\Delta a_i\|
   \]
   Similarly, $\exists x_2 \in \tilde{C}' \text{ s.t. } \text{dist}(x, x_2) \leq \max_i \|\Delta a_i\|
   
   Therefore, on the line from $x_1$ to $x_2$ \exists a point on $\text{bdry}(C')$ with distance from $x$ at most $\max_i \|\Delta a_i\|$

It takes a little more work to actually use this lemma to prove the fact that $K \geq r$.
Intuitively, what it means is that the boundary of the convex hull doesn’t move much by changing the $a_i$’s. Therefore, it is better to just change $c$.

In the next lecture, John Dunagan will do a smoothed analysis of the condition number. In particular, we shall prove the following theorem.

**Theorem 4** For $c, a_1, a_2, \ldots, a_n$ Gaussian random vectors with variance $\sigma^2$ and centered at $c, a_1, a_2, \ldots, a_n$, such that each has norm $\leq 1$,

\[
\Pr[K(c, a_1, a_2, \ldots, a_n) < \epsilon] \leq \frac{128d^{1/2} \epsilon}{\sigma}
\]

This result actually follows trivially from an earlier result, but with a less intuitive proof.

**Theorem 5 (Keith Ball ’93)** For any convex body $K$, and Gaussian random vector $c$,

\[
\Pr[\text{dist}(c, \text{bdry}(K)) < \epsilon] \leq \frac{8d^{3/4} \epsilon}{\sigma}
\]