1 Last time

We showed that $X \subset \mathbb{P}^n$ projective irreducible alg. set, then $X$ has structure of a prevariety, st. each $U_i \cap X \subset X$ are open and affine.

Also, we can think of $I \subset k[x_0, \ldots, x_n] \to R = \oplus_{n \geq 0} R_n$ a graded ring. Then $O_{X,x} = (R_{m_x})_0$. Similarly, for $U_{x_i} \subset \mathbb{P}^n$, we get $O_X(U_{x_i}) = (k[x_0, \ldots, x_n]_{x_i})_0 = k[x_1/x_i, \ldots, x_n/x_i]$.

We defined a morphism of prevarieties: if $X, Y$ are prevarieties then a morphism $X \to Y$ is a continuous map $f : X \to Y$ such that $\forall V \subset Y$ open and $\forall g \in \Gamma(V, O_Y)$ the composite $g \circ f$ is in $\Gamma(f^{-1}(V), O_X)$.

**Prop.** Suppose $X, Y$ are prevarieties, $f : X \to Y$ is continuous.

1. If there is a cover $Y = \bigcup V_i$ such that the restriction $f^{-1}(V_i) \to V_i$ is a morphism for every $i$ then $f$ is a morphism.

2. If there is a cover $X = \bigcup U_i$ such that $f|_{U_i} : U_i \to Y$ is a morphism for every $i$ then $f$ is a morphism.

**Pf.**

1. Let $V \subset Y$, $g \in \Gamma(V, O_Y)$. We have

$$\Gamma(f^{-1}(V), O_X) \to \prod_i \Gamma(f^{-1}(V \cap V_i), O_X) \to \prod_{i,j} \Gamma(f^{-1}(V \cap V_i \cap V_j), O_X)$$

This inclusion is due to the assumption, and by exactness this means that $g \circ f$ is actually in $\Gamma(f^{-1}(V), O_X)$ so $f$ is a morphism.

2. Same argument, but instead of using $f^{-1}(V \cap V_i)$ we use $f^{-1}(V) \cap U_i$.

2 Products of Varieties

Let $\mathcal{C}$ be a category, $X, Y \in Ob(\mathcal{C})$. Then a product of $X$ and $Y$ in $\mathcal{C}$ is a $Z$ such that $Z \to X$ and $Z \to Y$ such that if we have a map $W \to X$ and $W \to Y$ then there is a unique map $W \to Z$ such that all this commutes.

For example, $\mathcal{C} = Group$. Let $G_1, G_2$ be groups. Then the product of $G_1$ and $G_2$ is $G_1 \times G_2$ where multiplication is component-wise. If we have $H \to G_1$ and $H \to G_2$ then $H \to G_1 \times G_2$ defined by $\rho : h \mapsto (s(h), t(h))$.

Note: the direct sum is $G_1 \oplus G_2$; this has the complementary property; there are maps $G_1 \to G_1 \oplus G_2$ and $G_2 \to G_1 \oplus G_2$ such that if there are maps $G_1 \to H$ and $G_2 \to H$ then there is a unique map $G_1 \oplus G_2 \to H$ such that these all commute. In some instances, the direct sum and (direct) product are the same, but they are defined by these properties.

Another example: suppose we have an infinite set of vector spaces $\{V_i\}_{i=1}^{\infty}$, then there is an infinite product defined naturally: if there are maps from $Z$ to each $V_i$ then there is a unique map from $Z$ to the infinite product so that these maps compose with the “restriction” maps from the product to each $V_i$. In this case, the direct product is simply infinite vectors
where each component is from a particular $V_i$, while the direct sum is only such vectors where only finitely many are non-zero.

**Theorem.**

1. (Finite) products exist in the category of prevarieties.

2. If $X$ and $Y$ are affine varieties, then $X \times Y$ is an affine variety (in the product in the category of prevarieties: not what we proved on the homework).

3. If $X$ and $Y$ are projective varieties, then $X \times Y$ is a projective variety.

This may take some time to prove. Basically, we do this for the affine case, and then we do some gluing.

**Remark.** If it exists, as a set, $X \times Y$ is the product of the sets $X$ and $Y$.

If $X$ is a prevariety, then to give a point of $X$ is the same as giving a morphism $(*, k) \to X$ where $(*, k)$ is the variety of one point. When we consider morphisms $(*, k) \to X \times Y$ then this must give a morphism $(*, k) \to X$ and $(*, k) \to Y$. We write $|X|$ to denote the set of $X$. Then in general, $|X| = \text{Hom}((*, k)X)$. By the universal property,

$$|X \times Y| = \text{Hom}((*, k), X \times Y) \cong \text{Hom}((*, k), X) \times \text{Hom}((*, k), Y) = |X| \times |Y|,$$

2.1 Affine case

Try to do the affine case. Let $X, Y$ be affine. Let $R = \Gamma(X, O_X)$ and $S = \Gamma(Y, O_Y)$.

**Lemma.** $R \otimes_k S$ is a finitely generated integral domain if $R$ and $S$ are. It is enough to consider $R$, $S$ fields; we do this on HW5. It is enough because $R \otimes_k S$ can be thought of as sitting in $\text{Frac}(R) \otimes_k \text{Frac}(S)$; if this is an integral domain then $R \otimes_k S$ must be.

**Remark.** Define $Z$ to be the affine variety associated to $R \otimes_k S$.

**Lemma.** Let $X$ be a prevariety, $Y$ an affine variety. Then the map

$$\text{Hom}(X, Y) \to \text{Hom}(S, R)$$

is an isomorphism.

**Remark.** This proves that $Z = X \times Y$ is the product as prevarieties! If we have $W$ a prevariety, $T = \Gamma(W, O_W)$,

$$\text{Hom}(W, Z) \cong \text{Hom}(R \otimes S, T) \cong \text{Hom}(R, T) \times \text{Hom}(S, T) \cong \text{Hom}(W, X) \times \text{Hom}(W, Y).$$

**Pf.** of lemma. Say $X = \biguplus_i U_i$ where $U_i$ are affine open sets. Claim: the following is exact.
\[
\text{Hom}(X, Y) \rightarrow \prod_i \text{Hom}(U_i Y) \Rightarrow \prod_{i,j} \text{Hom}(U_i \cap U_j, Y).
\]

This follows from our proposition: clearly the image of the first is included in the kernel of the second; to show exactness, we say that it is enough to check continuousness on the \(U_i\)'s, and use our proposition.

Next, we have

\[
\begin{align*}
\text{Hom}(X, Y) & \rightarrow \prod_i \text{Hom}(U_i Y) \Rightarrow \prod_{i,j} \text{Hom}(U_i \cap U_j, Y) \\
\downarrow & \quad \downarrow \\
\text{Hom}(S, R) & \rightarrow \prod_i \text{Hom}(S, \Gamma(U_i, O_{U_i})) \Rightarrow \prod_{i,j} \text{Hom}(B, \Gamma(U_i \cap U_j, O_X))
\end{align*}
\]

The bottom sequence is exact (check this!). Then the first map (that is, \(\text{Hom}(X, Y) \rightarrow \text{Hom}(S, R)\)) is injective. The middle map is an isomorphism because the \(U_i\)'s are affine. The third map is also injective, and this proves that the first is an isomorphism by a nifty diagram chase. This proves the lemma.

### 2.2 General case

Now we will try to do the general case.

Let’s write \(|Z| = |X| \times |Y|\). For any \(U \subset X, V \subset Y\) affine open we can write \(|U \times V| \rightarrow Z\); if the \(U_i\) cover \(X\) and the \(V_i\) cover \(Y\) then the \(U_i \times V_j\) cover \(Z\).

**Claim:** To prove that \(X \times Y\) exists, it is enough to define a topology and sheaf of functions \(O_Z\) on \(|Z|\) such that \(\forall U \subset X, V \subset Y\) affine open sets, then \(|U \times V|\) is open and \((|U \times V|, O_Z|_{U \times V}) \cong U \times V\). This would make \((|Z|, O_Z)\) a prevariety; then to prove it is the product, we just have to construct the map \(W \rightarrow Z\) locally.

We’ll get there next time.