1 Completeness

Def. Let $X$ be a variety. We say $X$ is complete (proper) if for all $Y$ the map $X 	imes Y \to Y$ is closed.

Ex. $\mathbb{A}^n$ is not complete.

Lemma. (1) If $X$ and $Y$ are complete, $X \times Y$ is complete. (2) If $X$ is complete, $Z \subset X$ closed subvariety, then $Z$ is complete. (3) An affine variety $X$ is complete $\iff X = (\ast, k)$.

Pf. (1): Let $Z$ be a variety, we want $X \times Y \times Z \to Z$ to be closed. But this map is just the composition $X \times Y \times Z \to Y \times Z \to Z$, both of which are closed, so the map we want to be closed is closed.

(2): Let $Y$ be a variety, consider the map $Z \times Y \hookrightarrow X \times Y \to Y$. The first of these is closed (follows from the topology on products) and the second is closed by completeness of $X$.

Theorem 1. Any projective variety is complete.

Theorem 2. (Chow’s Lemma). If $X$ is a complete variety, then there is a projective variety $Y$ and a surjective birational map $\pi : Y \to X$.

We will get to the proof of these, hopefully today.

Lemma. If $f : X \to Y$ is a morphism of varieties and $X$ is complete, then $f(X) \subset Y$ is closed.

Proof. $f(x) = p_2(\Gamma_f(X))$ where $\Gamma_f$ is the graph $X \xrightarrow{\Gamma_f} X \times Y$ where $x \mapsto (x, f(x))$. We know this map is closed; it was part of the definition of varieties. Thus, $\Gamma_f(X)$ is closed, and $p_2$ is closed mapping $X \times Y \to Y$ since $X$ is complete, and so $f(X)$ is closed.

Completeness actually corresponds to the ability to complete maps to limit points. I.E. suppose we had a map $[0,1] \to X$, then we could complete the map to find one $[0,1] \to X$ that agrees.

Olsson takes an aside about the functor $h_X$. Basically, the above interpretation corresponds to the idea that $h_X(C) \to h_X(U)$ is surjective, where $C$ is dimension 1 and $U \to C$ where every local ring in $C$ is regular (i.e. $C = [0,1]$ and $U = [0,1]$.)

Pf. of Theorem 1.

Let $Y$ be a variety. We want $\mathbb{P}^n \times Y \to Y$ closed. We can assume $Y$ is affine. In fact we can assume $Y$ is $\mathbb{A}^{r}$ because we have the diagram

$$
\begin{array}{ccc}
\mathbb{P}^n \times Y & \to & Y \\
\downarrow & & \downarrow \\
\mathbb{P}^n \times \mathbb{A}^{r} & \to & \mathbb{A}^{r}
\end{array}
$$
Let $R = \Gamma(\mathbb{A}^r, O_{\mathbb{A}^r})$, and look at $S = R[X_0, \ldots, X_n]$. This is a graded ring. Closed
sets in $\mathbb{P}^n \times \mathbb{A}^r$ correspond to homogeneous ideals in $S$. To see this, let $U_i = U_{x_i \neq 0} \times \mathbb{A}^r$.
We have $\Gamma(U_i, O_{U_i}) = \Gamma[I_0, \ldots, x_n^n] \cong R \otimes_k R = (S_{x_i})_{(0)}$, the degree $0$ part of $S$ localized at $x_i$. Now think about the intersection of any closed set with the open cover, and we get a homogeneous ideal.

Note here: it really isn’t doing us any good here to be using $\mathbb{A}^r$ instead of an arbitrary affine variety.

Let $I(Z)$ be the ideal generated by homog. $f \in S$ such that $f(Z) = 0$.

**Lemma.** For all $i$, $I(Z \cap U_i)$ is generated by the image of $(I(Z)_{x_i})_{(0)} \to I(Z \cap U_i)$.

**Pf.** of lemma. Let $g \in I(Z \cap U_i) \subset R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. An element in this ring, it has
some denominator, so we can multiply by some power of $x_i$ to get $x_i^mg \in R[x_0, \ldots, x_n]$.
This may not vanish everywhere on $Z$, for instance on $Z \cap (U_i^c) = Z \cap V(x_i)$, because when
$x_i = 0$ this may not work, so we just multiply by one more $x_i$ to get $x_i^mg$ and this now vanishes on $Z \cap U_i$.

So we have a homogeneous $A \subset S, V(A) = Z$, and we have our map $\mathbb{P}^n \times \mathbb{A}^n \to \mathbb{A}^n$. Choose $y \notin p_2(Z)$. We will show there is an open set including $y$ which has empty
intersection with $p_2(Z)$; this will prove that $p_2(Z)^c$ is open, and thus prove $p_2(Z)$ is closed.

Let $m \subset \mathcal{R}$ be the maximal ideal of $y$. Consider $Z \cap U_i \cap U_{x_i \neq 0} \times Y \subset U_{x_i \neq 0} \times \{y\}$.
Both are closed maps, and they do not intersect.

We have $m \mapsto R \to R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]/I(Z \cap U_i)$. We know $m$ maps to the ideal (1) since this
corresponds to the maps above, in which the intersection is empty. Thus, $1 = a_i + \sum_j m_{i,j}g_i$,
where $a_i \in I(Z \cap U_i)$ and $m_{i,j} \in m$ in $R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Thus, there is some $N_i$ such that
$x_i^{N_i} = a_i^m + \sum_j m_{i,j}g_i$ where $a_i^m \in I(Z)$ and this equation holds in $S$.

Look at the degree $N_i$ piece $S_{N_i} = (R[X_0, \ldots, X_n])_{N_i}$. So we get $S_N = A_N + mS_N$ for
some big enough $N$. Thus, $M = S_N/A_N$ is a f.g. module over $R$, and $mM = M$ so there
is an $f \in R - m$ such that $fS_N \subset A_N$ by Nakayama’s lemma.

This $f$ is the one we want. Consider $D(f)$, $D(f) \cap p_2(Z) = \emptyset$. This is because $fS_N \subset A_N \Rightarrow p_2(V(A)) \subset V(f) \subset Y$. This completes the proof of Theorem 1.

**Prop.** $f : \mathbb{P}^m \to \mathbb{P}^m$, $W = f(\mathbb{P}^m)$ is a variety. Then either dim $W = n$ or dim $W = 0$. It
may not be an isomorphism: We had our map $t \mapsto t'^2, t'^3 - t'^2$ (I think? The graph he drew
looks like an $a$) and the cusp has two preimages.

**Pf.** Let $r = \text{dim } W$, assume $1 \leq r \leq n - 1$. We know there is a list $f_1, \ldots, f_{r+1} \in k[X_0, \ldots, X_m]$ such that $W \cap V((f_1, \ldots, f_{r+1})) = \emptyset$ and $W \cap V(f_j) \neq \emptyset$. (This is applying
that we can keep intersecting with $V(f_j)$, each time reducing the dimension by 1, until we get
down to the emptyset.)

Let $Z_i = f^{-1}(V(f_j))$. We know $Z_1 \cap \ldots \cap Z_{r+1} = \emptyset$. There are two possibilities. Either
$Z_i = \mathbb{P}^n$ or they’re a hypersurface. We know $r + 1 \leq n$, but this can’t happen! We proved
this before: the intersection of $\leq n$ hypersurfaces is nonempty in $\mathbb{P}^n$.

Thus each $Z_i = \mathbb{P}^n$, but then their intersection can’t be empty because there is at least
1 of them ($r \geq 1$). Thus, we have a contradiction, so the dimension of $W$ is either 0 or $n$. 
2 Complex Topology

Martin wants to talk about complex topology for a bit, and then about curves.

Let $X \subset \mathbb{A}^n_{\mathbb{C}}$. What is the complex topology? Take the usual topology on $\mathbb{C}^n$ and give $X$ the induced topology. There is a sheaf of rings $O_X$ on $X$ here; we define this as $O_{\mathbb{C}^n}(U) = \{ \text{holomorphic functions } U \to \mathbb{C} \}$. If $V \subset X_{an}$ is open, then define $O_{X_{an}}(V) = \{ f : V \to \mathbb{C} \text{ such that } \forall v \in V \text{ there is a neighborhood } U \text{ of } v \text{ in } \mathbb{A}^n \text{ and } \hat{f} \in O_{\mathbb{C}^n}(U) \text{ restricting to } f \}$. NOTE: $X_{an}$ is basically $X$ under the induced topology from $\mathbb{C}^n$.

**Def.** An *analytic space* is a pair $(X, O_X)$ where $X$ is a top. space, $O_X$ is a sheaf of functions $X \to \mathbb{C}$ such that there is a finite open cover $U_i$ of $X$ such that $(U_i, O_X|_{U_i})$.

Whoops, out of time.