Lecture 9: Chow’s Lemma, Blowups

Last time we showed that projective varieties are complete. The following result from Wei-Liang Chow gives a partial converse. Recall that a birational morphism between two varieties is an isomorphism on some pair of open subsets.

**Lemma 1** (Chow’s Lemma). *If X is a complete, irreducible variety, then there exists a projective variety \( \hat{X} \) that is birational to X.*

**Proof.** This proof is a standard one. Here we follow the proof presented by [SH77]. Choose an affine covering \( X = U_1 \cup \ldots \cup U_n \), and let \( Y_i \supseteq U_i \) be projective varieties containing \( U_i \) as open subsets. Now consider \( \Delta : U \to U^n = \coprod U_i \to Y \) where \( U = \bigcap U_i, Y = \coprod Y_i \), and \( \phi : U \to X \times Y \) be induced by the standard inclusion \( U \to X \) and \( \Delta \). Let \( \hat{X} \) be the closure of \( \phi(U) \), and \( \pi_1 \) gives a map \( \hat{f} : \hat{X} \to X \). This map is birational because \( f^{-1}(U) = \phi(U) \), and on \( U \) the map \( \pi_1 \circ \phi \) is just identity. (To see the first claim, note that it means \( (U \times Y) \cap \hat{X} = \phi(U) \), i.e. \( \phi(U) \) is closed in \( U \times Y \), which is true because \( \phi(U) \) is \( U \times Y \) is just the graph of \( \Delta \), which is closed as \( Y \) is separated.)

So it remains to check that \( \hat{X} \) is projective. We show this by showing that the restriction of \( \pi_2 : X \times Y \to Y \) to \( \hat{X} \), which we write as \( g : \hat{X} \to Y \), is a closed embedding. Let \( V_i = \pi_i^{-1}(U_i) \), where \( \pi_i \) is the projection map from \( Y \) to \( Y_i \). First we claim that \( \pi_2^{-1}(V_i) \) cover \( \hat{X} \), which easily follow from the statement that \( \pi_2^{-1}(V_i) = f^{-1}(U_i) \), since \( U_i \) cover \( X \). Consider \( W = f^{-1}(U) = \phi(U) \) as an open subset in \( f^{-1}(U_i) \): on \( W \) we have \( f = p_i g \), so the same holds on \( f^{-1}(U_i) \) and the covering property follows.

It remains to show that \( \hat{X} \cap V_i \to U_i \) are closed embeddings. Noting that \( V_i = Y_1 \times \ldots \times Y_{i-1} \times U_i \times Y_{i+1} \times \ldots \times Y_n \), we write \( Z_i \) to denote the graph of \( V_i \to U_i \), and note that it is closed and isomorphic to \( V_i \) via projection. Noting that \( \phi(U) \subseteq Z_i \) and that \( Z_i \) is closed, taking closure we see that \( \hat{X} \cap V_i \to U_i \) is closed in \( Z_i \).

**Blowing up of a point in \( \mathbb{A}^n \)** The blow-up of the affine \( n \)-space at the origin is defined as \( \hat{\mathbb{A}}^n = Bl_0(\mathbb{A}^n) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} = \{(x, L) : x \in \mathbb{A}^n, L \in \mathbb{P}^{n-1}, x \in L \} \). It is a variety defined by equations \( x_i t_j = x_j t_i \). We have a projection \( \pi : \hat{\mathbb{A}}^n \to \mathbb{A}^n \). Atop 0 there is an entire \( \mathbb{P}^{n-1} \), and on the remaining open set the projection is an isomorphism.

Now consider \( X \) an closed subset of \( \mathbb{A}^n \), such that \( \{0\} \) is not a component. The **proper transform** of \( X \) (a.k.a. the blowup of \( X \) at 0), denoted \( \hat{X} \), is the closure of the preimage of \( X \setminus \{0\} \) under \( \pi \). Suppose \( X \) contains 0, then \( \pi^{-1}(X) = \hat{X} \cup \mathbb{P}^{n-1} \). If \( X \subseteq \mathbb{A}^n \), then \( \mathbb{P}^{n-1} \subseteq \hat{X} \) because \( \dim(\mathbb{P}^{n-1}) = \dim(\hat{X}) \). If \( X \) is irreducible, then \( \hat{X} \) is the irreducible component of \( \pi^{-1}(X) \) other than \( \mathbb{P}^{n-1} \). The preimage of 0 within \( \hat{X} \) is called the **exceptional locus**.

Next, observe that \( \hat{\mathbb{A}}^n \) is covered by \( n \) affine charts. More explicitly, \( \hat{\mathbb{A}}^n_i \subseteq \mathbb{A}^{n-1} \times \mathbb{A}^n \) has coordinates \( (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \). On there, the defining equation becomes \( x_j t_i = x_j t_i \) for \( j \neq i \), so \( \hat{\mathbb{A}}^n_i \cong \mathbb{A}^n \) with coordinates \( (t_1, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n) \). In other words, if \( P(x_1, \ldots, x_n) \subseteq I_X \), then \( P(t_1 x_1, \ldots, t_{i-1} x_i, x_i, \ldots) \subseteq I_{X \cap \hat{\mathbb{A}}^n_i} \).

**Example 1.** Let \( X = (y^2 = x^3 + x^2) \subseteq \mathbb{A}^2 \). Suppose \( y = tx \), then \( t^2 x^2 = x^3 + x^2 \implies t^2 = x + 1 \), so the preimage of \((0,0)\) is \( \{t = \pm 1, x = 0\} \). Thus \( X \) is not normal because the map \( \hat{X} \to X \) is not 1-to-1, though \( \deg(\hat{X} \to X) = 1 \) (recall that a finite birational morphism to a normal variety is isomorphism).

**Definition 1.** Let \( X \) an affine variety, \( x \in X \), we write \( Bl_x(X) = \hat{X}_x \) to denote \( \hat{X} \) for an embedding \( X \subseteq \mathbb{A}^n \) where \( x \to 0 \).

**Remark 1.** \( Bl_x(X) \) contains \( X \setminus x \) as an open set, so this generalizes to any variety \( X \).

**Proposition 1.** Suppose \( X \) embeds via two embeddings \( i_1, i_2 \) to \( \mathbb{A}^n \) and \( \mathbb{A}^m \) respectively, such that there exists some \( x \) such that \( i_1(x) = i_2(x) = 0 \), then \( X_1 = \hat{X}_2 \) for two blowups at \( x \).

In particular, this tells us that blowup is an intrinsic operation that does not depend on the embedding.
Proof. First consider the special case $X = \mathbb{A}^n$, $i_1 = \text{id}$, and $i_2$ given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f)$ for some polynomial $f$. Write $\mathbb{A}^{n+1} = \bigcup_{i=1}^{n+1} \mathbb{A}^n_i$, and observe that $\bigcup_{i=1}^n \mathbb{A}^{n+1}_i \setminus \{0 : \ldots : 0 : 1\} \subset \mathbb{P}^n$.

Call that point $\infty$, then one can check that $\infty \notin \mathbb{A}^n$. Now note that $\mathbb{A}^n \cap \mathbb{A}^{n+1}_{i} = \mathbb{A}^n \subseteq \mathbb{A}^{n+1}$ (Locally write it as $t_{n+1}x_i = f(t_1x_i, \ldots, x_i, \ldots, t_nx_i)$, and observe we have a $x_i$ on both sides so the closure would be of shape $t_{n+1} = f'(t_1, \ldots, x_i, \ldots, t_n)$, which gives an entire $\mathbb{A}^n$), so together we see that the blowup is nothing but $\mathbb{A}^n$. Second, consider $X = \mathbb{A}^n$, $i_1 = \text{id}$, $i_2 : \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$ being a graph of a morphism $\mathbb{A}^n \to \mathbb{A}^m$.

This can be reduced to the first case by induction on $m$ (or really, just the exactly same argument applied several times). Now consider the general case of arbitrary $i_1, i_2$. First extend the embedding $i_2 : X \to \mathbb{A}^m$ to a map $\mathbb{A}^n \to \mathbb{A}^m$ by lifting each generator (one can switch to the algebraic side, suppose $X = \text{Spec } A$, then we get two surjective maps $\psi_1 : k[x_1, \ldots, x_m] \to A$ and $\psi_2 : k[y_2, \ldots, y_n] \to A$, lift $\psi_1$ to $\psi_2 \circ \phi$ for $\phi : k[x_1, \ldots, x_m] \to k[y_1, \ldots, y_n]$ where we map each $x_i$ into $A$, then lift), then one can use part 2.

$(x \mapsto i_1(x)) \mapsto \infty$ has the same blowup as $x \mapsto i_1(x) \mapsto (i_1(x), i_2(x))$, which has the same blowup as $x \mapsto i_2(x) \mapsto i_2(x)$ by the same argument applied on the other direction.)

As an application, consider an example of a complete non-projective surface: start with $\mathbb{P}^1 \times \mathbb{P}^1$, blow it up at $(0,0)$, consider the projection to the second factor. For any $x \neq 0$, the preimage of $x$ is a projective line; for $x = 0$, the preimage is the union of two projective lines (one can see this by passing to affine chart then consider closure). Consider two copies of this blow up, call them $X, Y$, and call the two exceptional lines $L_1, L_2$ for both of them. Now consider the disjoint union of $X$ and $Y$ where we identify $L_1$ of $X$ with the fiber of $\infty$ of $Y$, and vise versa.

References
