18.781 Practice Questions for Midterm 2

Note: The actual exam will be shorter (about 10 of these questions), in case you are timing yourself.

1. Find a primitive root modulo $343 = 7^3$.
2. How many solutions are there to $x^{12} \equiv 7 \pmod{19}$? To $x^{12} \equiv 6 \pmod{19}$?
3. Solve the congruence $3x^2 + 4x - 2 \equiv 0 \pmod{31}$.
4. Characterize all primes $p$ such that $15$ is a square modulo $p$.
5. If $n$ is odd, evaluate the Jacobi symbol $\left( \frac{n^3}{n-2} \right)$.
6. If $n = p_1^{e_1} \ldots p_r^{e_r}$, how many squares modulo $n$ are there? How many quadratic residues modulo $n$ are there (i.e. the squares which are coprime to $n$)?
7. Let $p > 3$ be a prime. Show that the number of solutions $(x, y)$ of the congruence $x^2 + y^2 \equiv 3 \pmod{p}$ is $p - \left( \frac{-1}{p} \right)$.
8. Compute (with justification) the cyclotomic polynomial $\Phi_{12}(x)$.
9. Let $f(n) = (-1)^n$. Compute

$$Z(f, 2) = \sum_{n \geq 1} \frac{f(n)}{n^2}.$$ 

(you may use that $\sum 1/n^2 = \pi^2/6$.)

10. For $n = p_1^{e_1} \ldots p_r^{e_r}$, calculate the value of $(U * U * U)(n)$, where $U$ is the arithmetic function such that $U(n) = 1$ for all $n$.

11. Let $p$ be a prime which is 1 mod 4, and suppose $p = a^2 + b^2$ with $a$ odd and positive. Show that $\left( \frac{a}{p} \right) = 1$.

12. Let $a_1, a_2, a_3, a_4$ be integers. Show that the product $p = \prod_{i<j} (a_i - a_j)$ is divisible by 12.

13. Let the sequence $\{a_n\}$ be given by $a_0 = 0, a_1 = 1$ and for $n \geq 2$,

$$a_n = 5a_{n-1} - 6a_{n-2}.$$ 

Show that for every prime $p > 3$, we have $p \mid a_p$.

14. Find a positive integer such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$.

15. Compute the set of integers $n$ for which $\sum_{d \mid n} \mu(d)\phi(d) = 0$.

16. Let $f$ be a multiplicative function which is not identically zero. Show that $f(1) = 1$. 

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