18.786 Problem Set 7 (due Thursday Apr 8 in class)

1. Let $L, M$ be finite extensions of a field $K$.
   (a) If $L, M$ are Galois over $K$, then so is their compositum $LM$.
   (b) If $L, M$ are Abelian over $K$, so is $LM$.
   (c) If $K$ is a number field, $p$ a prime of $\mathcal{O}_K$ which is unramified in $L$ and $M$, then $p$ is unramified in $LM$.

2. Prove that $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p$. (Hint: use unique factorization and the Chinese remainder theorem.)

3. Let $L/K$ be a finite extension of number fields. Let $v$ be an absolute value on $K$ (archimedean or nonarchimedean), and let $K_v$ be the completion of $K$ with respect to $v$.
   (a) Show that every extension $w$ to $L$ of the valuation $v$ arises as the composite $\bar{v} \circ \tau$ for some $K$-embedding $\tau : L \to \overline{K_v}$ into the algebraic closure of $K_v$ (here $\bar{v}$ is the unique extension of $v$ to the algebraic closure), and that two such extensions $\bar{v} \circ \tau$ and $\bar{v} \circ \tau'$ are equal iff $\tau$ and $\tau'$ are conjugate over $K_v$.
   (b) Show that $L \otimes K_v \cong \prod_{w|v} L_w$, where the product is over all valuations $w$ which extend $v$. When $v$ is non-archimedean corresponding to the prime $p$ of $\mathcal{O}_K$, the $w$ are in one-to-one correspondence with the primes $\mathfrak{P}$ lying above $p$. (Hint: Use Proposition 2 of Samuel, section 5.2 to show this.)
   (c) If $L/K$ is Galois then show that all the extensions are conjugates. For $G = Gal(L/K)$, let $G_w = \{g \in G \mid gw = w\}$. Show that $L_w$ is Galois over $K_v$ and $G_w$ is its Galois group.

4. Let $K$ be a nonarchimedean local field and $\mathcal{O}_K$ its valuation ring. Let $U = \mathcal{O}_K^\times$ be the units of $\mathcal{O}_K$. Endow $\mathcal{O}_K$ and $U$ with the metric/topology induced from the valuation on $K$. Show that $U$ is compact, and open and closed in $\mathcal{O}_K$. Show that a subgroup of the additive group $\mathcal{O}_K$ is open iff it is of finite index, and the same statement for the multiplicative group $U$.

5. Show that cubic field $K$ generated over $\mathbb{Q}$ by a root of $x^3 - x^2 - 2x - 8$ is not monogenic. (Hint: figure out how 2 splits in $K$, and argue by contradiction.)

6. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. Show that $\mathbb{C}_p$ is algebraically closed. (Hint: use Krasner’s lemma.)