Chapter 14

Isotherman Parameters

Let \( x : U \to S \) be a regular surface. Let

\[
\phi_k(z) = \frac{\partial x_k}{\partial u_1} - i \frac{\partial x_k}{\partial u_2}, \quad z = u_1 + i u_2.
\]

(14.1)

Recall from last lecture that

(a) \( \phi \) is analytic in \( z \) \( \iff \) \( x_k \) is harmonic in \( u_1 \) and \( u_2 \).

(b) \( u_1 \) and \( u_2 \) are isothermal parameters \( \iff \)

\[
\sum_{k=1}^{n} \phi_k^2(z) = 0 \quad (14.2)
\]

c) If \( u_1, u_2 \) are isothermal parameters, then \( S \) is regular \( \iff \)

\[
\sum_{k=1}^{n} |\phi_k(z)|^2 \neq 0 \quad (14.3)
\]

We start by stating a lemma that summarizes what we did in the last lecture:

**Lemma 4.3 in Osserman:** Let \( x(u) \) define a minimal surface, with \( u_1, u_2 \) isothermal parameters. Then the functions \( \phi_k(z) \) are analytic and they satisfy the eqns in b) and c). Conversely if \( \phi_1, \phi_2, ..., \phi_n \) are analytic functions satisfying the eqns in b) and c) in a simply connected domain \( D \)
then there exists a regular minimal surface defined over domain $D$, such that the eqn on the top of the page is valid.

Now we take a surface in non-parametric form:

$$x_k = f_k(x_1, x_2), k = 3, ..., n$$ (14.4)

and we have the notation from the last time:

$$f = (f_3, f_4, ..., f_n), p = \frac{\partial f}{\partial x_1}, q = \frac{\partial f}{\partial x_2}, r = \frac{\partial^2 f}{\partial x_1^2}, s = \frac{\partial^2 f}{\partial x_1 \partial x_2}, t = \frac{\partial^2 f}{\partial x_2^2}$$ (14.5)

Then the minimal surface eqn may be written as:

$$(1 + |q|^2)\frac{\partial p}{\partial x_1} - (p.q)(\frac{\partial p}{\partial x_2} + \frac{\partial q}{\partial x_1}) + (1 + |p|^2)\frac{\partial q}{\partial x_2} = 0$$ (14.6)

equivalently

$$(1 + |q|^2)r - 2(p.q)s + (1 + |p|^2)t = 0$$ (14.7)

One also has the following:

$$detg_{ij} = 1 + |p|^2 + |q|^2 + |p|^2|q|^2 - (p.q)^2$$ (14.8)

Define

$$W = \sqrt{detg_{ij}}$$ (14.9)

Below we’ll do exactly the same things with what we did when we showed that the mean curvature equals 0 if the surface is minimizer for some curve. Now we make a variation in our surface just like the one that we did before (the only difference is that $x_1$ and $x_2$ are not varied.)

$$\tilde{f}_k = f_k + \lambda h_k, k = 3, ..., n,$$ (14.10)

where $\lambda$ is a real number, and $h_k \in C^1$ in the domain of definition $D$ of the
We have
\[ \tilde{f} = f + \lambda h, \tilde{p} = p + \lambda \frac{\partial h}{\partial x_1}, \tilde{q} = q + \lambda \frac{\partial h}{\partial x_2} \] (14.11)

One has
\[ \tilde{W}^2 = W^2 + 2\lambda X + \lambda^2 Y \] (14.12)

where
\[ X = [(1 + |q|^2)p - (p.q)q] \frac{\partial h}{\partial x_1} + [(1 + |p|^2)q - (p.q)p] \frac{\partial h}{\partial x_2} \] (14.13)

and \( Y \) is a continuous function in \( x_1 \) and \( x_2 \). It follows that
\[ \tilde{W} = W + \lambda \frac{X}{W} + O(\lambda^2) \] (14.14)

as \( |\lambda| \to 0 \). Now we consider a closed curve \( \Gamma \) on our surface. Let \( \Delta \) be the region bounded by \( \Gamma \). If our surface is a minimizer for \( \Delta \) then for every choice of \( h \) such that \( h = 0 \) on \( \Gamma \) we have
\[ \int \int_{\Delta} \tilde{W} dx_1 dx_2 \geq \int \int_{\Delta} W dx_1 dx_2 \] (14.15)

which implies
\[ \int \int_{\Delta} \frac{X}{W} = 0 \] (14.16)

Substituting for \( X \), integrating by parts, and using the fact that \( h = 0 \) on \( \Gamma \), we find
\[ \int \int_{\Delta} \left[ \frac{\partial}{\partial x_1} \left[ \frac{1 + |q|^2}{W} p - \frac{p.q}{W} q \right] + \frac{\partial}{\partial x_2} \left[ \frac{1 + |p|^2}{W} q - \frac{p.q}{W} p \right] \right] h dx_1 dx_2 = 0 \] (14.17)

must hold everywhere. By the same reasoning that we used when we found the condition for a minimal surface the above integrand should be zero.
\[ \frac{\partial}{\partial x_1} \left[ \frac{1 + |q|^2}{W} p - \frac{p.q}{W} q \right] + \frac{\partial}{\partial x_2} \left[ \frac{1 + |p|^2}{W} q - \frac{p.q}{W} p \right] = 0 \] (14.18)
Once we found this equation it makes sense to look for ways to derive it from the original equation since after all there should only be one constraint for a minimal surface. In fact the LHS of the above eqn can be written as the sum of three terms:

\[
\left[ 1 + |q|^2 \frac{\partial p}{W} \frac{\partial x_1}{\partial x_1} - \frac{p \cdot q}{W} \frac{\partial q}{\partial x_1} + \frac{\partial p}{\partial x_2} + \frac{1 + |p|^2}{W} \frac{\partial q}{\partial x_2} \right] + \left[ \frac{\partial}{\partial x_1} \left( \frac{1 + |q|^2}{W} \right) - \frac{\partial}{\partial x_2} \left( \frac{p \cdot q}{W} \right) \right] p \tag{14.19} \\
+ \left[ \frac{\partial}{\partial x_2} \left( \frac{1 + |p|^2}{W} \right) - \frac{\partial}{\partial x_1} \left( \frac{p \cdot q}{W} \right) \right] q \tag{14.20}
\]

The first term is the minimal surface eqn given on the top of the second page. If we expand out the coefficient of \( p \) in the second term we find the expression:

\[
\frac{1}{W^3} \left[ (p \cdot q) q - (1 + |q|^2)p \right] \cdot [(1 + |q|^2)r - 2(p \cdot q)s + (1 + |p|^2)t] \tag{14.22}
\]

which vanishes by the second version of the minimal surface eqns. Similarly the coefficient of \( q \) in third term vanishes so the while expression equals zero. In the process we’ve also shown that

\[
\frac{\partial}{\partial x_1} \left( \frac{1 + |q|^2}{W} \right) = \frac{\partial}{\partial x_2} \left( \frac{p \cdot q}{W} \right) \tag{14.23}
\\
\frac{\partial}{\partial x_2} \left( \frac{1 + |p|^2}{W} \right) = \frac{\partial}{\partial x_1} \left( \frac{p \cdot q}{W} \right) \tag{14.24}
\]

**Existence of isothermal parameters or Lemma 4.4 in Osserman**

Let \( S \) be a minimal surface. Every regular point of \( S \) has a neighborhood in which there exists a reparametrization of \( S \) in terms of isothermal parameters.

**Proof:** Since the surface is regular for any point there exists a neighborhood of that point in which \( S \) may be represented in non-parametric form. In particular we can find a disk around that point where the surface can be
represented in non parametric form. Now the above eqns imply the existence of functions $F(x_1, x_2)$, $G(x_1, x_2)$ defined on this disk, satisfying

$$\frac{\partial F}{\partial x_1} = \frac{1 + |p|^2}{W}, \frac{\partial F}{\partial x_2} = \frac{p \cdot q}{W};$$  \hfill (14.25)

$$\frac{\partial G}{\partial x_1} = \frac{p \cdot q}{W}, \frac{\partial G}{\partial x_2} = \frac{1 + |q|^2}{W}.$$ \hfill (14.26)

If we set

$$\xi_1 = x_1 + F(x_1, x_2), \xi_2 = x_2 + G(x_1, x_2),$$ \hfill (14.27)

we find

$$J = \frac{\partial(x_1, x_2)}{\partial(x_1, x_2)} = 2 + \frac{2 + |p|^2 + |q|^2}{W} \geq 0$$ \hfill (14.28)

Thus the transformation $(x_1, x_2) \rightarrow (\xi_1, \xi_2)$ has a local inverse $(\xi_1, \xi_2) \rightarrow (x_1, x_2)$. We find the derivative of $x$ at point $(\xi_1, \xi_2)$:

$$Dx = J^{-1}[x_1, x_2, f_3, ..., f_n]$$ \hfill (14.29)

It follows that with respect to the parameters $\xi_1, \xi_2$ we have

$$g_{11} = g_{22} = \left| \frac{\partial x}{\partial \xi_1} \right|^2 = \left| \frac{\partial x}{\partial \xi_2} \right|^2$$ \hfill (14.30)

$$g_{12} = \frac{\partial x}{\partial \xi_1} \cdot \frac{\partial x}{\partial \xi_2} = 0$$ \hfill (14.31)

so that $\xi_1, \xi_2$ are isothermal coordinates.