Chapter 4

Implicit Function Theorem

4.1 Implicit Functions

Theorem 4.1.1. Implicit Function Theorem Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuously differentiable in an open set containing $(a, b)$ and $f(a, b) = 0$. Let $M$ be the $m \times m$ matrix $D_{n+j}^i f(a, b), 1 \leq i, j \leq m$ If $\det(M) \neq 0$, there is an open set $A \subset \mathbb{R}^n$ containing $a$ and an open set $B \subset \mathbb{R}^m$ containing $b$, with the following property: for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function $g$ is differentiable.

proof Define $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. Then $\det(dF(a, b)) = \det(M) \neq 0$. By inverse function theorem there is an open set $W \subset \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a, b) = (a, 0)$ and an open set in $\mathbb{R}^n \times \mathbb{R}^m$ containing $(a, b)$, which we may take to be of the form $A \times B$, such that $F : A \times B \rightarrow W$ has a differentiable inverse $h : W \rightarrow A \times B$. Clearly $h$ is the form $h(x, y) = (x, k(x, y))$ for some differentiable function $k$ (since $f$ is of this form) Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be defined by $\pi(x, y) = y$; then $\pi \circ F = f$. Therefore $f(x, k(x, y)) = f \circ h(x, y) = (\pi \circ F) \circ h(x, y) = \pi(x, y) = y$ Thus $f(x, k(x, 0)) = 0$ in other words we can define $g(x) = k(x, 0)$

As one might expect the position of the $m$ columns that form $M$ is immaterial. The same proof will work for any $f'(a, b)$ provided that the rank
of the matrix is $m$.

**Example** $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 + y^2 - 1$. $Df = (2x\, 2y)$ Let $(a, b) = (3/5, 4/5)$ $M$ will be $(8/5)$. Now implicit function theorem guarantees the existence and the uniqueness of $g$ and open intervals $I, J \subset \mathbb{R}, 3/5 \in I, 4/5 \in J$ so that $g : I \to J$ is differentiable and $x^2 + g(x)^2 - 1 = 0$. One can easily verify this by choosing $I = (-1, 1), J = (0, 1)$ and $g(x) = \sqrt{1-x^2}$. Note that the uniqueness of $g(x)$ would fail to be true if we did not choose $J$ appropriately.

**example** Let $A$ be an $m \times (m+n)$ matrix. Consider the function $f : \mathbb{R}^{n+m} \to \mathbb{R}^m, f(x) = Ax$ Assume that last $m$ columns $C_{n+1}, C_{n+2}, \ldots, C_{m+n}$ are linearly independent. Break $A$ into blocks $A = [A' | M]$ so that $M$ is the $m \times m$ matrix formed by the last $m$ columns of $A$. Now the equation $AX = 0$ is a system of $m$ linear equations in $m+n$ unknowns so it has a nontrivial solution. Moreover it can be solved as follows: Let $X = [X_1 | X_2]$ where $X_1 \in \mathbb{R}^{n \times 1}$ and $X_2 \in \mathbb{R}^{m \times 1}$ $AX = 0$ implies $A'X_1 + MX_2 = 0 \Rightarrow X_2 = M^{-1}A'X_1$. Now treat $f$ as a function mapping $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ by setting $f(X_1, X_2) = AX$. Let $f(a, b) = 0$. Implicit function theorem asserts that there exist open sets $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ and a function $g : I \to J$ so that $f(x, g(x)) = 0$. By what we did above $g = M^{-1}A'$ is the desired function. So the theorem is true for linear transformations and actually $I$ and $J$ can be chosen $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

### 4.2 Parametric Surfaces

(Following the notation of Osserman $E^n$ denotes the Euclidean $n$-space.) Let $D$ be a domain in the $u$-plane, $u = (u_1, u_2)$. A parametric surface is simply the image of some differentiable transformation $u : D \to E^n$. (A non-empty open set in $\mathbb{R}^2$ is called a domain.)

Let us denote the Jacobian matrix of the mapping $x(u)$ by
\[ M = (m_{ij}); m_{ij} = \frac{\partial x_i}{\partial u_j}, i = 1, 2, ..., n; j = 1, 2. \]

We introduce the exterior product

\[ v \wedge w; w \wedge v \in E^{n(n-1)/2} \]

where the components of \( v \wedge w \) are the determinants \( \det \begin{pmatrix} v_i & v_j \\ u_i & u_j \end{pmatrix} \) arranged in some fixed order. Finally let

\[ G = (g_{ij}) = M^T M; g_{ij} = \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \]

Note that \( G \) is a \( 2 \times 2 \) matrix. To compute \( \det(G) \) we recall Lagrange’s identity:

\[
\left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right) - \left( \sum_{k=1}^n a_k b_k \right)^2 = \sum_{1 \leq i, j \leq n} (a_i b_j - a_j b_i)^2
\]

Proof of Lagrange’s identity is left as an exercise. Using Langrange’s identity one can deduce

\[
\det(G) = \left| \frac{\partial x}{\partial u_1} \wedge x u_2 \right|^2 = \sum_{1 \leq i, j \leq n} \left( \frac{\partial (x_i, x_j)}{\partial (u_1, u_2)} \right)^2
\]