Chapter 5

First Fundamental Form

5.1 Tangent Planes

One important tool for studying surfaces is the tangent plane. Given a given regular parametrized surface $S$ embedded in $\mathbb{R}^n$ and a point $p \in S$, a tangent vector to $S$ at $p$ is a vector in $\mathbb{R}^n$ that is the tangent vector $\alpha'(0)$ of a differential parametrized curve $\alpha: (-\epsilon, \epsilon) \rightarrow S$ with $\alpha(0) = p$. Then the tangent plane $T_p(S)$ to $S$ at $p$ is the set of all tangent vectors to $S$ at $p$. This is a set of $\mathbb{R}^3$-vectors that end up being a plane.

An equivalent way of thinking of the tangent plane is that it is the image of $\mathbb{R}^2$ under the linear transformation $Dx(q)$, where $x$ is the map from a domain $D \rightarrow S$ that defines the surface, and $q$ is the point of the domain that is mapped onto $p$. Why is this equivalent? We can show that $x$ is invertible.

So given any tangent vector $\alpha'(0)$, we can look at $\gamma = x^{-1} \circ \alpha$, which is a curve in $D$. Then $\alpha'(0) = (x \circ \gamma)'(0) = (Dx(\gamma(0)) \circ \gamma')(0) = Dx(q)(\gamma'(0))$. Now, $\gamma$ can be chosen so that $\gamma'(0)$ is any vector in $\mathbb{R}^2$. So the tangent plane is the image of $\mathbb{R}^2$ under the linear transformation $Dx(q)$.

Certainly, though, the image of $\mathbb{R}^2$ under an invertible linear transformation (it’s invertible since the surface is regular) is going to be a plane including the origin, which is what we’d want a tangent plane to be. (When
I say that the tangent plane includes the origin, I mean that the plane itself consists of all the vectors of a plane through the origin, even though usually you’d draw it with all the vectors emanating from \( p \) instead of the origin.)

This way of thinking about the tangent plane is like considering it as a “linearization” of the surface, in the same way that a tangent line to a function from \( \mathbb{R} \to \mathbb{R} \) is a linear function that is locally similar to the function. Then we can understand why \( D_x(q)(\mathbb{R}^2) \) makes sense: in the same way we can “replace” a function with its tangent line which is the image of \( \mathbb{R} \) under the map \( t \mapsto f'(p)t + C \), we can replace our surface with the image of \( \mathbb{R}^2 \) under the map \( D_x(q) \).

The interesting part of seeing the tangent plane this way is that you can then consider it as having a basis consisting of the images of \( (1,0) \) and \( (0,1) \) under the map \( D_x(q) \). These images are actually just (if the domain in \( \mathbb{R}^2 \) uses \( u_1 \) and \( u_2 \) as variables) \( \frac{\partial x}{\partial u_1} \) and \( \frac{\partial x}{\partial u_2} \) (which are \( n \)-vectors).

### 5.2 The First Fundamental Form

Nizam mentioned the First Fundamental Form. Basically, the FFF is a way of finding the length of a tangent vector (in a tangent plane). If \( w \) is a tangent vector, then \( |w|^2 = w \cdot w \). Why is this interesting? Well, it becomes more interesting if you’re considering \( w \) not just as its \( \mathbb{R}^3 \) coordinates, but as a linear combination of the two basis vectors \( \frac{\partial x}{\partial u_1} \) and \( \frac{\partial x}{\partial u_2} \). Say \( w = a \frac{\partial x}{\partial u_1} + b \frac{\partial x}{\partial u_2} \); then

\[
|w|^2 = \left( a \frac{\partial x}{\partial u_1} + b \frac{\partial x}{\partial u_2} \right) \cdot \left( a \frac{\partial x}{\partial u_1} + b \frac{\partial x}{\partial u_2} \right) = a^2 \frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_1} + 2ab \frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2} + b^2 \frac{\partial x}{\partial u_2} \cdot \frac{\partial x}{\partial u_2}. \tag{5.1}
\]

Let’s deal with notational differences between do Carmo and Osserman. do Carmo writes this as \( Ea^2 + 2Fab + Gb^2 \), and refers to the whole thing as \( I_p : T_p(S) \to \mathbb{R} \).\(^1\) Osserman lets \( g_{11} = E \), \( g_{12} = g_{21} = F \) (though he never

\(^1\)Well, actually he’s using \( u' \) and \( v' \) instead of \( a \) and \( b \) at this point, which is because these coordinates come from a tangent vector, which is to say they are the \( u'(q) \) and \( v'(q) \)
makes it too clear that these two are equal), and $g_{22} = G$, and then lets the matrix that these make up be $G$, which he also uses to refer to the whole form. I am using Osserman’s notation.

Now we’ll calculate the FFF on the cylinder over the unit circle; the parametrized surface here is $x: (0, 2\pi) \times \mathbb{R} \to S \subset \mathbb{R}^3$ defined by $x(u, v) = (\cos u, \sin u, v)$. (Yes, this misses a vertical line of the cylinder; we’ll fix this once we get away from parametrized surfaces.) First we find that $rac{\partial x}{\partial u} = (-\sin u, \cos u, 0)$ and $rac{\partial x}{\partial v} = (0, 0, 1)$. Thus $g_{11} = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial u} = \sin^2 u + \cos^2 u = 1$, $g_{21} = g_{12} = 0$, and $g_{22} = 1$. So then $|w|^2 = a^2 + b^2$, which basically means that the length of a vector in the tangent plane to the cylinder is the same as it is in the $(0, 2\pi) \times \mathbb{R}$ that it’s coming from.

As an exercise, calculate the first fundamental form for the sphere $S^2$ parametrized by $x: (0, \pi) \times (0, 2\pi) \to S^2$ with

$$x(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). \quad (5.2)$$

We first calculate that $\frac{\partial x}{\partial \theta} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$ and $\frac{\partial x}{\partial \varphi} = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0)$. So we find eventually that $|w|^2 = a^2 + b^2 \sin^2 \theta$. This makes sense — movement in the $\varphi$ direction (latitudinally) should be “worth more” closer to the equator, which is where $\sin^2 \theta$ is maximal.

### 5.3 Area

If we recall the exterior product from last time, we can see that $|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}|$ is the area of the parallelogram determined by $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$. This is analogous to the fact that in 18.02 the magnitude of the cross product of two vectors is the area of the parallelogram they determine. Then $\int_Q |\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}| \, du \, dv$ is the area of the bounded region $Q$ in the surface. But Nizam showed yesterday
that Lagrange’s Identity implies that
\[
\left| \frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} \right|^2 = \left| \frac{\partial x}{\partial u} \right|^2 \left| \frac{\partial x}{\partial v} \right|^2 - \left( \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} \right)^2
\] (5.3)

Thus \( \frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = \sqrt{g_{11}g_{22} - g_{12}^2} \). Thus, the area of a bounded region \( Q \) in the surface is \( \int_Q \sqrt{g_{11}g_{22} - g_{12}^2} \, du \, dv \).

For example, let us compute the surface area of a torus; let’s let the radius of a meridian be \( r \) and the longitudinal radius be \( a \). Then the torus (minus some tiny strip) is the image of \( x: (0, 2\pi) \times (0, 2\pi) \to S^1 \times S^1 \) where \( x(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v), r \sin u \). Then \( \frac{\partial x}{\partial u} = (-r \sin u \cos v, -r \sin u \sin v, r \cos u) \), and \( \frac{\partial x}{\partial v} = (- (a + r \cos u) \sin v, (a + r \cos u) \cos v, 0) \). So \( g_{11} = r^2, g_{12} = 0, \) and \( g_{22} = (r \cos u + a)^2 \). Then \( \sqrt{g_{11}g_{22} - g_{12}^2} = r(r \cos u + a) \). Integrating this over the whole square, we get

\[
A = \int_0^{2\pi} \int_0^{2\pi} (r^2 \cos u + ra) \, du \, dv
\]
\[
= \left( \int_0^{2\pi} (r^2 \cos u + ra) \, du \right) \left( \int_0^{2\pi} \, dv \right)
\]
\[
= (r^2 \sin 2\pi + ra 2\pi)(2\pi) = 4\pi^2 ra
\]

And this is the surface area of a torus!

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