Lecture 10
Regularized Pricing and Risk Models
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Mr. Masyukov’s comments today are his own, and do not necessarily represent the views of Morgan Stanley or its affiliates, and are not a product of Morgan Stanley Research.
Plan for today

- Bonds
- Swaps
- Yield curve
- Regularized yield curve models
- Regularized volatility surface
Bonds

- A debt security
- Borrower issues bonds to obtain funds
- Investor purchases bond to earn return
- Typical bonds include fixed periodic coupon payments plus face value at maturity
- Zero coupon bonds – only face value at maturity, no coupons
- There are perpetual bonds – infinite regular coupon payments, but no face value, as the bonds never mature
Bond Cashflows

- Fixed rate bonds (periodic coupon payments and principal at maturity)
- Zero coupon bond
- Sum of future cashflows is not equal to bond price because future cashflows are less valuable (time value of money)
- Discount factor

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Bond Price

- Present price of the bond should be the sum of present values (PV) of future cashflows

\[ P = \sum_{i=1}^{N} cF\Lambda_i + F\Lambda_N \]

Where
- \( P \) – fair bond price
- \( F \) – face value of bond
- \( \Lambda_i \) - discount factor for payment date \( i \)
- \( c \) – coupon rate
- \( N \) – number of coupon periods

- Need model for discounting \( \Lambda_i \)

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Yield to Maturity

- Use one parameter \( y \) – yield to maturity to compute all discount factors

\[
\Lambda_i = e^{-yt_i}
\]

\[
P = e^{-yt_1}cF + e^{-yt_2}cF + \ldots + e^{-yt_N}cF + e^{-yt_N}F
\]

\[
P = \sum_{i=1}^{N} e^{-yt_i}C_i
\]

Where
- \( y \) – yield to maturity
- \( t \) – future time of payment, years
- \( C_i \) - \( i \)-th cashflow

- Continuous compounding case
- Assumed constant \( y \) for all \( t_i \)

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Bond Duration

• Sensitivity of bond price \((\ln(P))\) to bond yield

\[
d = \frac{1}{P} \frac{\partial P}{\partial y}
\]

\[
d = -\frac{1}{P} \sum_{i=1}^{N} t_i e^{-y_i} C_i = -\frac{\sum_{i=1}^{N} t_i e^{-y_i} C_i}{\sum_{i=1}^{N} e^{-y_i} C_i}
\]

Where \(d \) – bond duration \(C_i\) - \(i\)-th cashflow

• Duration = “weighted time”

• Duration of zero coupon bond always equals to its maturity

• Duration of regular coupon bond is always less then its maturity

• As there is just one \(y\) for all payment dates, the duration is a sensitivity to “parallel” move
Bond Convexity

- Second derivative of bond price to bond yield

\[ c = \frac{\partial^2 P}{\partial y^2} \]

\[ d = \sum_{i=1}^{N} t_i^2 e^{-yt_i} C_i \]

Where  \( c \) – bond convexity  
\( C_i \) - \( i \)-th cashflow

- Duration is good measure for price changes for small variation in yield
- Second derivative needed for large changes in yields
- Convexity is always positive

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Fixed-vs-float swap analytics

- Valuing fixed and float legs of the swap

\[ PV_{fixed} = \sum_i C_i \delta_i \Lambda_i = C \sum_i w_i \]
\[ PV_{float} = \sum_i r_i \delta_i \Lambda_i = \sum_i r_i w_i \]
\[ PV_{fixed} = PV_{float} \]
\[ C = \frac{\sum_i r_i w_i}{\sum_i w_i} \]

Where

\( C \) – Swap rate (fixed leg coupon)
\( \Lambda_i \) - discount factor for payment date \( i \)
\( \delta_i \) – day count fraction
\( r_i \) – forward rate (floating rate of future payment)

- Swap rate is weighted sum of forward rates (assumed same frequency of payments of fixed and floating legs)

- Swap can be hedged with bond
Constructing Yield Curve

- Select input instruments
- Choose interpolation
  - Interpolation space (daily forward rates, zero rates, etc.)
  - Spline (piecewise-constant, linear, tension spline, etc.)
  - Knot points and model parameters
- Calibrate = solve for spline parameters such that input instruments are re-priced at par
Yield Curve Graph

- Graph of 3M forward rates
Bond Spread to Yield Curve

• We have curve now. So we can use can compute more accurate discount factors $\Lambda_i$, rather than relying on “flat” curve with same $y$ for all cashflow dates

• Need extra parameter bond spread $s$ to match with bond price

$$P = \sum_{i=1}^{N} e^{-st_i} \Lambda_i C_i$$

Where $\Lambda_i$ - discount factor for payment date $i$ computed from the curve
$s$ – bond spread
$t_i$ – future time of payment, years
$C_i$ - $i$-th cashflow

• If model is available for typical movements of the curve embedded in $\Lambda_i$ we can build more effective risk model for bond, rather than using single “parallel” shift mode (bond duration)
Shifting 9Y swap by 1 basis point

- Response of 3M forward rates
## Portfolio Risk and Cost of Hedging

### Portfolio risk and Bid-Offer charge per bucket

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Quote</th>
<th>Raw Risk</th>
<th>B/O charge bp</th>
<th>Charge</th>
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Hedging Portfolio risks - Formulation

\[ x = \arg \min \{ \| F^T (r + Hx) \|^2 \} \]

- **r** – portfolio risk
- **H** – hedging portfolio risks
- **x** – weights of hedging instruments
- **F** – market scenarios (factors)
Principal Component Analysis (PCA)

$$D = USP^T$$

- Use SVD to decompose market movements data $D$ into principal components $P$ and corresponding uncorrelated market dynamics $U$ with weights $S$
- Use few SVD components with largest singular values - low rank approximation of market data
- Principal components $P$ are eigen vectors of covariance matrix $D^TD$
Main Principal Components of Swap Rates

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Hedging Portfolio Risks - PCA

\[ P^T (r + Hx) = 0 \]
\[ x = (P^T H)^{-1} P^T r \]
\[ x = R^T r \]
\[ R = P(H^T P)^{-1} \]

- **P** – PCA factors
- **H** – risk of hedging portfolio (liquid swaps)
- **R** – risk transform matrix

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### Hedging using PCA model

#### PCA Matrix

<table>
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<tr>
<th>Swap</th>
<th>Raw Risk</th>
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<th>2Y</th>
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#### PCA risk

- (426,757)
- 1,892,770
- (1,538,808)
- (268,764)
- 293,261

#### B/O charge

- 0.1
- 0.1
- 0.1
- 0.1
- 0.1

#### Charge

- 42,676
- 189,277
- 153,881
- 26,876
- 29,326

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PCA Risk Model

- “Formally” tuned to historical data
- Hedge coefficients are not stable, especially if historical widow is short to reflect recent regime
- Costly to re-hedge when PC factors change
- Instability is coming from noisy PCs corresponding to small singular values
- Over-fitting to historical data
- No assumptions used about shape of the yield curve

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PCA Interpretation

- Risk matrix $\mathbf{R}$ is linear combinations $\mathbf{Y}$ of principal components $\mathbf{P}$ producing shifts of one hedging instrument at a time

$$
\mathbf{R} = \mathbf{P}\mathbf{Y}
$$

$$
\mathbf{H}^T\mathbf{R} = \mathbf{I}
$$

$$
\mathbf{R} = \mathbf{P}(\mathbf{H}^T\mathbf{P})^{-1}
$$

- Can we build risk model $\mathbf{R}$ based on some reasonable assumption (such as smoothness of forward rates) rather than purely historical data?
Regularized Risk Model

• Assumption: Forward rates move smoothly

\[ H^T R = I \]
\[ \| LJR \|^2 \rightarrow \text{min} \]
\[ R \sim (HH^T + \lambda^2 (LJ)^T LJ)^{-1} \]

• Where J – Jacobian matrix translating shifts of yield curve inputs to movements of forward rates, L – smoothness regularity matrix, \( \lambda \) - small regularization parameter
## Regularized Model Risk Projection

<table>
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<tr>
<th>Swap</th>
<th>Raw Risk</th>
<th>1Y</th>
<th>2Y</th>
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Heath-Jarrow-Morton (HJM) Model

- Evolution of forward rates

\[
df_{t,s} = \mu_{t,s} \, dt + \int_{t,s} \beta V(t,s) \rho(t,s) \cdot dB_t^Q
\]

- \( f \) – forward rate
- \( \mu \) – drift
- \( \beta \) – model skew factor
- \( \rho \) – correlation/factor structure
- \( V(t,s) \) – parametric volatility surface (our main focus today)
- \( dB_t^Q \) – Brownian motion

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Forward Rates Map

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Volatile Surface Calibration Challenge

- High dimensionality (need to calibrate ~28k elements)
- No memory to store 28k X 28k matrix
- Relatively small number of calibration instruments (20-50)
- Under-determined problem
- Sensitivity areas of calibration instruments overlap significantly
- Ill posed inverse problem
- Unstable, noisy solution
- Need regularity constraints
- Has to be smooth to produce realistic prices for similar instruments
Formal Approach to Calibration

- Represent volatility surface as a linear combination of $N$ basis functions
  \[ v = v_0 + B \cdot x \]
  
  - $v$ – vector containing elements of the volatility grid
  - $B$ – matrix, columns corresponding to basis functions
  - $x$ – vector of weights

- Make $N$ equivalent to the number of calibration instruments $M$
- “Formally” unambiguous
- Make basis functions piecewise constant matching sensitivity of calibration instruments, 0 otherwise
Compute sensitivities (Jacobian matrix)

- Use pricing model to compute sensitivities of prices of calibration instruments to perturbations of volatility surface

\[
J_{ij} = \frac{\partial q_i}{\partial x_j}
\]

\[
q = J \cdot x
\]

\[
q = \ln \frac{q_{mdl}}{q_0}, q_{in} = \ln \frac{q_{market}}{q_0}
\]

Where \( J \) – Jacobian matrix

\( q_{mdl}, q_{market}, q_0 \) – model, market, and base price

\( x \) – vector of basis functions coefficients
Solve

- $J$ is square and invertible, as basis functions are selected reasonably
- Iteratively solve for basis function coefficients $\mathbf{x}$

$$q_{in} = J \cdot \mathbf{x}$$

$$\mathbf{x} = J^{-1}q_{in}$$

- Quickly converges, as (typically) price is $\sim$proportional to volatility for at-the-money calibration instruments
“Formal” solution

- Exact, but ... meaningless
Attempt to improve solution

- Using smoothed piece-wise constant basis functions
Calibration problem: Ill-posed

- 1% change of input price of 5y x 10y swaption results in 4% change of vol surface adjustment

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Key Improvements of Calibration

• Use ill-posedness to our advantage:
  - Allow some tolerance to calibration accuracy of input instruments
  - Significant improvements in the (smoothness of) output surface may not cost much in terms of accuracy of calibration
  - Calibration instruments have different liquidity, and bid-offer spread. So we can use weights to decrease tolerance for important instruments

• Basis functions:
  - Absolutely need basis functions to reduce dimensionality of the inverse problem
  - Need many (M > N instruments) basis functions, as we do not know in advance which shapes will work
  - 2-dimensional B-Splines
B-Spline representation

Cubic spline

B-spline basis functions

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Building B-splines using Cox-de Boor recursion formula

B-spline basis functions, order 1

B-spline basis functions, order 2

B-spline basis functions, order 3

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2-d B-spline functions
Formulate the problem

\[ \| W \cdot (q - q_{in}) \|^2 \rightarrow \min \]
\[ \| L_1 \cdot (v - v_0) \|^2 \rightarrow \min \]
\[ \| L_2 \cdot v \|^2 \rightarrow \min \]

Where \( W \) – diagonal matrix of weights
\( L_1 \) – regularization matrix for change
\( L_2 \) – regularization matrix for result

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Surface Gradient Penalty

Example of regularization matrix $L_1$

$$L_1 =
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & -1 & 0 & 0
\end{bmatrix}$$

No investment decisions should be made in reliance on this material.
Solution of Regularized Optimization Problem

\[ x = \arg \min \left\{ \| W(Jx - q_{in}) \|^2 + \| \lambda_1 L_1 Bx \|^2 + \| \lambda_2 L_2 x \|^2 \right\} \]

\[ x = A \cdot J^T W^2 q_{in}, \text{where} \]

\[ A = (J^T W^2 J + \lambda_1^2 (L_1 B)^T L_1 B + \lambda_2^2 L_2^T L_2)^{-1} \]

Where \( L_2 \) – Tikhonov regularization matrix
Calibration Result

No investment decisions should be made in reliance on this material.
Calibration Inverse Problem

\[ y = Ax + \varepsilon \]
\[ x = (A^T A)^{-1} A^T y \]
\[ A = USV^T = \sum_i s_i u_i v_i^T \]
\[ U^T U = V^T V = I \]
\[ x = VS^{-1} U^T y = \sum_i \frac{u_i^T y}{s_i} v_i \]

- \( y \) – market inputs, \( x \) – model parameters
- Singular Value Decomposition of (forward) model matrix \( A \)
- \( s_i \) – singular values
- Result: Rotation \( \rightarrow \) Scaling \( 1/s_i \) \( \rightarrow \) Rotation

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Ill Posed Problem

\[ x = \sum_{i} \frac{u_i^T \epsilon}{s_i} v_i \]

- Input noise \( \epsilon \) may be magnified by small singular values \( s_i \)
- Condition number \( \max(s_i)/\min(s_i) \) as indicator of ill-posedness
- Small variation in input results in large change in the solution
“Noiseless” situation

\[ x = \sum_{i} \frac{u_i^T y}{s_i} v_i \]

\[ p = Bx \]

\[ p_y = Ax = A(A^T A)^{-1} A^T y = y \]

- We compute \( x \) to calculate price of the portfolio \( p \).
- If all singular values are non-zero, we “formally” re-price inputs, as \( A^T(A^T A)^{-1} A = I \) – identity. So, the model appears to be accurate.
- However, pricing of actual portfolio \( p \) with model \( B \) may be unstable.
Tikhonov Regularization

\[
x = \arg \min \{ \| Ax - y \|^2 + \| \lambda x \|^2 \}
\]

\[
x = (A^T A + \lambda^2 I)^{-1} A^T y
\]

\[
x = V \frac{S}{S^2 + \lambda^2 I} U^T y = \sum_i w_i \cdot \frac{u_i^T y}{s_i} v_i
\]

\[
w_i = \frac{s_i^2}{s_i^2 + \lambda^2}
\]

- Apply penalty to amplitude of model parameters \( x \)
- Re-pricing matrix \( A^T(A^T A + \lambda^2 I)^{-1} A \neq I \) is no longer 100% accurate, however
- More stable model vector \( x \), and pricing of actual portfolio
Truncated SVD (TSVD)

\[
x = \sum_{i} w_i \cdot \frac{u_i^T y}{s_i} v_i
\]

\[w_i = 1, i \leq N\]
\[w_i = 0, i > N\]

- Truncation of effective rank of the model matrix \(A\)
- Similarity with PCA (principal component) approach
- Truncated is “null” space of the model: parameter modes, which do not affect calibration accuracy
- The problem is when “null” space of the model has noticeable impact on portfolio pricing
Regularized Models

- Improved stability
- Regularization is essential for ill-conditioned problems
- More realistic solution at the expense of fitting input data
- May cause bias to the solution
- Bias can be minimized by proper selection of the penalty constraints
Useful Links

- PCA: http://en.wikipedia.org/wiki/Principal_component_analysis
- B-Splines: http://en.wikipedia.org/wiki/B-spline
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