Recall Moment-Curvature Equation

\[ M(x) = \frac{E(x)I(x)}{\rho(x)} \text{ or } EI_{\text{eff}} \text{ for composite beams.} \]

\[ \frac{1}{\rho(x)} = \frac{\partial^2 v}{\partial x^2} = \frac{M(x)}{E(x)I(x)} \]

Approach: Integrate, get \( v(x), \theta(x) = \frac{dv(x)}{dx} \). Use boundary conditions to get constants of integration.

Library of Solutions (Thus Far):

\[ v(x) = \frac{Mx^2}{2EI} \]

\[ v(x) = \frac{-Px^2}{2EI} \left( L - \frac{x}{3} \right) \]

\[ v_{\text{tip}}(x = L) = \frac{-PL^3}{3EI} \]

Recall example from last time:
\[ M(x) = P \left( 1 - \frac{a}{L} \right) x, 0 \leq x \leq a \]

\[ M(x) = P \left( 1 - \frac{a}{L} \right) x - P \left( x - a \right), a \leq x \leq L \]

For \( a = \frac{L}{2} \) (symmetric special case)

\[ M(x) = \frac{Px}{2}, 0 \leq x \leq \frac{L}{2} \]

\[ M(x) = -\frac{Px}{2} + \frac{PL}{2} \frac{L}{2} \leq x \leq L \]

Left:

\[ \frac{d^2 v}{dx^2} = \frac{P_x}{2EI} \]

\[ \frac{dv}{dx} = \frac{P_x^2}{4EI} + c_1 \]

\[ v = \frac{P_x^3}{12EI} + c_1 x + c_2 \]

Boundary Conditions: \( v(x = 0) = 0 \Rightarrow c_2 = 0 \)

\( \theta(x = \frac{L}{2}) = 0 \)

\[ 0 = P \frac{L^2}{4EI} \left( \frac{L}{2} \right)^2 + c_1 \Rightarrow c_1 = -\frac{PL^2}{16EI} \]

So:

Left:

\[ v(x) = \frac{P}{12EI} x^3 - \frac{PL^2}{16EI} x, 0 \leq x \leq \frac{L}{2} \]

Right:
\[ v(x) = \frac{P}{12EI} (L - x)^3 - \frac{PL^2}{16EI} (L - x), 0 \leq x \leq \frac{L}{2} \]

Note: The right is the same as the left but starting at \( x = L \) and moving left. This is due to symmetry. This situation is called 3-point bending.

One can define a stiffness in "F=kx" type equation.

Find \( v_{\text{max}} \)

\[ v\left(\frac{L}{2}\right) = \frac{P}{EI} \left[ \frac{1}{12} \left(\frac{L}{2}\right)^3 - \frac{1}{16} \left(\frac{L}{2}\right)^2 \right] \]

\[ v_{\text{max}} = \frac{PL^3}{64EI} \left[ \frac{1}{3} - 1 \right] \]

\[ v_{\text{max}} = -\frac{PL^3}{48EI} \]

\( F = kx \)

\( -P = kv_{\text{max}} \)

So: \( k = \frac{48EI}{L^3} \)

Now solve again without using symmetry:

Recall:

\[ v_L(x) = \frac{P x^3}{12EI} + c_1 x + c_2, 0 \leq x \leq \frac{L}{2} \]

Recall:

\[ M_z(x) = -\frac{P x}{2} + \frac{P L}{2}, \frac{L}{2} \leq x \leq L \]

\[ \frac{d^2 v_r}{dx^2} = \frac{1}{EI} \left[ -\frac{P x}{2} + \frac{P L}{2} \right] \]
\[
\theta_R(x) = \frac{dv_r}{dx} = \frac{1}{EI} \left[ \frac{-P x^2}{4} + \frac{P L x}{2} \right] + c_3 \\
v_r = \frac{1}{EI} \left[ \frac{-P x^3}{12} + \frac{P L x^2}{4} \right] + c_3 x + c_4
\]

Boundary Conditions

1. \( v_L(0) = 0 \)
2. \( v_R(L) = 0 \)
3. \( v_L \left( \frac{L}{2} \right) = v_R \left( \frac{L}{2} \right) \)
4. \( \theta_L \left( \frac{L}{2} \right) = \theta_R \left( \frac{L}{2} \right) \)

Use 1.

\[ v_L(0) = 0 \]
\[ c_2 = 0 \]

Use 2.

\[ v_R(L) = 0 \]
\[ \frac{P}{EI} \left[ \frac{-L^3}{12} + \frac{L^3}{4} \right] + c_3 L + c_4 \]
\[ c_3 L + c_4 = \frac{-PL^3}{6EI} \]

Use 3.

\[ v_L \left( \frac{L}{2} \right) = v_R \left( \frac{L}{2} \right) \]
\[ \frac{P}{EI} \left[ \frac{1}{12} \left( \frac{L}{2} \right)^3 \right] + c_1 \frac{L}{2} = \frac{P}{EI} \left[ \frac{-1}{12} \left( \frac{L}{2} \right)^3 + \frac{L^3}{16} \right] + c_3 \frac{L}{2} + c_4 \]
\[ (c_3 - c_1) \frac{L}{2} + c_4 = \frac{PL^3}{EI} \left( \frac{-1}{24} \right) \]

Use 4.

\[ \theta_L \left( \frac{L}{2} \right) = \theta_R \left( \frac{L}{2} \right) \]
\[ c_3 - c_1 = \frac{-PL^2}{8EI} \]

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Solve for $c_1, c_2, c_3$.
Note: Applying symmetry was easier.

Example: Statically Indeterminate

FBD

Solve using superposition
1. Pretend $R_{By}$ is known.

2. Find $v(x)$ and $v_{\text{tip}}$.

$$v(x) = \frac{M}{2EI} x^2 + \frac{R_{By} x}{2EI} \left( L - \frac{x}{3} \right)$$

$$v_{\text{tip}} = \frac{ML^2}{2EI} + \frac{R_{By} L^3}{3EI}$$

3. Note $v_{\text{tip}} = 0$ due to support. This is an additional boundary condition.
\[
\frac{ML^2}{2EI} + \frac{R_B y L^3}{3EI} = 0
\]

\[R_B y = -\frac{3M}{2L}\]

4. Solve for \(v(x)\).

\[
v(x) = \frac{M}{2EI}x^2 - \frac{3M}{2L} \left( \frac{x^2}{2EI} \right) \left( L - \frac{x}{3} \right)
\]

Discontinuity Functions

\(< x - a > \equiv 0 \text{ for } x - a < 0\)

\(< x - a > \equiv x - z \text{ for } x - a > 0\)

\[
\int < x - a >^n \, dx = \frac{< x - a >^{n+1}}{n+1} + c
\]

So recall example:

\[
M_z(x) = P \left( 1 - \frac{a}{L} \right) x - P < x - a >
\]

\[
\frac{d^2v(x)}{dx^2} = \frac{1}{EI} \left[ P \left( 1 - \frac{a}{L} \right) x - P < x - a > \right]
\]
\[
\frac{dv(x)}{dx} = \frac{1}{EI} \left[ P \left( 1 - \frac{a}{L} \right) x - P < x - a > \right] + c_1
\]

\[
v(x) = \frac{1}{EI} \left[ P \left( 1 - \frac{a}{L} \frac{x^3}{6} - \frac{P < x - a >^3}{6} \right) \right] + c_1 x + c_2
\]

Boundary Conditions:
\[v(0) = 0 \Rightarrow c_2 = 0\]
\[v(L) = 0 \Rightarrow c_1 = \frac{P}{EI} \left( \left( 1 - \frac{a}{L} \right) \frac{L^3}{6} - \frac{(L-a)^3}{6} \right)\]

So:
\[
v(x) = \frac{P}{EI} \left[ \left( 1 - \frac{a}{L} \right) \frac{x^3}{6} - \frac{< x - a >^3}{6} \right] - \frac{P}{EI} \left[ \left( 1 - \frac{a}{L} \right) \frac{L^3}{6} - \frac{(L-a)^3}{6} \right] x
\]

So:
\[
v_L(x) = \frac{P}{EI} \left[ \left( 1 - \frac{a}{L} \right) \frac{x^3}{6} \right] - \frac{P}{EI} \left[ \left( 1 - \frac{a}{L} \right) \frac{L^3}{6} - \frac{(L-a)^3}{6} \right] x
\]
\[
v_R(x) = \frac{P}{EI} \left[ \left( 1 - \frac{a}{L} \right) \frac{x^3}{6} - \frac{(x-a)^3}{6} \right] - \frac{P}{EI} \left[ \left( 1 - \frac{a}{L} \right) \frac{L^3}{6} - \frac{(L-a)^3}{6} \right] \frac{x}{L}
\]

Check answer. Try \(a = \frac{L}{2}\) and compare to earlier result.
\[
v \left( \frac{L}{2} \right) = \frac{-PL^3}{48EI}
\]