Lecture 6
Moderately Large Deflection Theory of Beams

Problem 6-1:
Part A: The department of Highways and Public Works of the state of California is in the process of improving the design of bridge overpasses to meet earthquake safety criteria. As a highly paid consultant to the project, you were asked to evaluate its soundness. You rush back to your lecture notes, and you model the overpass as a simply supported beam of span $L$ with an overhang $\Delta=0.01L$. Assume that the distributed load is a sinusoidal function.

\[
\frac{EI}{EA} \quad N \quad L \quad \Delta
\]

a) Calculate the maximum allowable midspan deflection $(w_0)_{critical}$ under which the beam will slide off its support.

Part B: Assume that the above design with an external axial force $N=0$ and $\Delta=0.01L$ has a safety factor of one. The design of earthquake resistant structures requires a safety factor of five, meaning that $(w_0)_{critical}$ must be increased by a factor of five without the bridge collapsing. Two possible design modifications were proposed. In the first one, the overhang is simply increased to $\Delta_{new}$. In the second design, a tensile force $N$ is applied to the bridge to increase its transverse stiffness and thus reduce the central deflection and the resulting motion of the support.

b) For the first proposed modification, what length $\Delta_{new}$ of the overhang will meet the requirement of a safety factor of five? Give your result in terms of the original $\Delta$ and other parameters if needed.

c) For the second design, what is the magnitude of the dimensionless tensile force $N/EA$ that will give a safety factor equal to five?

d) Which design is better? Can you think of a third alternative design solution?
**Problem 6-1 Solution:**

Recall:

\[
\frac{NL}{EA} = -\Delta + \int_0^L \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \, dx
\]

(a) Calculate the max deflection \( w_o \) at \( L/2 \) under which the beam will slide off its support

Assume \( N = 0 \)

\[
\Delta = \int_0^L \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \, dx \tag{1}
\]

Since the applied load is a sin function, we can assume the deflected shape will also be a sine function

\[
w(x) = w_o \sin\left( \frac{\pi x}{L} \right)
\]

\[
w'(x) = w_o \left( \frac{\pi}{L} \right) \cos\left( \frac{\pi x}{L} \right)
\]

\[
(w'(x))^2 = w_o^2 \left( \frac{\pi}{L} \right)^2 \cos^2\left( \frac{\pi x}{L} \right)
\]

Substitute the above equation into equation (1), we have

\[
\Delta = \int_0^L \frac{1}{2} w_o^2 \left( \frac{\pi}{L} \right)^2 \cos^2\left( \frac{\pi x}{L} \right) \, dx
\]

\[
= \frac{1}{2} w_o^2 \left( \frac{\pi}{L} \right)^2 \left[ \frac{x}{2} + \frac{\sin(\pi x/L)}{4\pi/L} \right]_0^L
\]

\[
= \frac{1}{2} w_o^2 \left( \frac{\pi}{L} \right)^2 \left[ \frac{L}{2} + 0 \right]
\]

\[
= \frac{1}{L} \left( \frac{w_o \pi}{2} \right)^2
\]

\[
\Delta = \frac{1}{L} \left( \frac{w_o \pi}{2} \right)^2
\]
Thus, the maximum allowable midspan deflection is

\[
W_o^2 = \frac{4L\Delta}{\pi^2}
\]

\[
\boxed{W_o = \frac{2}{\pi}\sqrt{L\Delta}}
\]

We’re given \( \Delta = 0.01L \)

\[
W_o = \frac{2}{\pi}\sqrt{\frac{L^2}{100}} = \frac{L}{5\pi}
\]

\[
\boxed{W_o = \frac{L}{5\pi}}
\]

(b) Case 1: increase \( \Delta \) to meet safety factor of 5

\[
5W_o = \frac{2}{\pi}\sqrt{L\Delta_{new}}
\]

\[
\Delta_{new} = \frac{1}{L}\left(\frac{5W_o \pi}{2}\right)^2
\]

\[
\Delta_{new} = \frac{1}{L}\left(\frac{5W_o \pi}{2}\right)^2 = 25\left[\frac{1}{L}\left(\frac{w_o \pi}{2}\right)^2\right]
\]

Recall: \( \Delta = \frac{1}{L}\left(\frac{w_o \pi}{2}\right)^2 \)

\[
\boxed{\Delta_{new} = 25\Delta}
\]

(c) Case 2: apply a tensile force \( N \), so now \( N \neq 0 \)

\[
\frac{NL}{EA} = -\Delta + \int_0^L \frac{1}{2} \left( \frac{dw}{dx} \right)^2 dx
\]

\[
\frac{NL}{EA} = -\Delta + \frac{1}{L} \left( \frac{w_o \pi}{2} \right)^2
\]
We want to calculate $N$ that will give us the equivalent effect of applying a safety factor of 5, in which case

$$\frac{1}{L} \left( \frac{w_{s,\pi}}{2} \right)^2 = 25\Delta$$

$$\frac{NL}{EA} = -\Delta + 25\Delta = 24\Delta$$

In the case of $\Delta = L/100$

$$\frac{N}{EA} = \frac{24}{L \cdot \frac{100}{100}} = 0.24$$

(d) Which design is better?

It is difficult to say which design is better. Each design has its advantage and disadvantages. For case 1, we will have a long overhang which may not be aesthetically pleasing. For case 2, it may be difficult to apply constantly a tensile force.

Other options include a stiffer simply supported beam, or add cables to suspend the bridge.
Problem 6-2:
A long span aerial tramway steel cable of length \( L = 1 \text{km} \) is loaded by a hurricane wind with intensity \( q(x) \) sinusoidally distributed between the end stations. The cable deflects by \( w_0 = 5 \text{m} \).

\[
\begin{align*}
E &= 2.1 \times 10^5 \text{ MPa} \\
\sigma_y &= 300 \text{ MPa} \\
D &= 60 \text{ mm}
\end{align*}
\]

Cross-section of cable

\[
q(x) = q_0 \sin \left( \frac{\pi x}{L} \right)
\]

\( q(x) \)

\( w_0 \)

\( q_0 \)

\( L \)

\( a) \) Calculate the resulting load intensity \( q_0 \)

\( b) \) Calculate the tension in the cable \( N \).

\( c) \) Calculate the tensile stress.

\( d) \) Compare (c) with the yield stress, and determine the safety factor.

Problem 6-2 Solution:
Using the equation of equilibrium

\[
EIw'' - Nw'' = q
\]

However, a cable has no bending stiffness, so our equation becomes:

\[
-Nw'' = q
\]

\[
w'' = -\frac{q}{N} = -\frac{1}{N} q_0 \sin \left( \frac{\pi x}{L} \right)
\]

Integrate twice

\[
w' = \frac{q_0}{N} \left( \frac{L}{\pi} \right) \cos \left( \frac{\pi x}{L} \right) + C_1
\]

\[
w = \frac{q_0}{N} \left( \frac{L}{\pi} \right)^2 \sin \left( \frac{\pi x}{L} \right) + C_1 x + C_2
\]
Plug in the boundary conditions to solve for the constants
\[
\begin{aligned}
\begin{cases}
w(0) = 0 & \Rightarrow C_2 = 0 \\
w(L) = 0 & \Rightarrow C_1L + C_2 = 0 \Rightarrow C_1 = 0 \\
\end{cases}
\end{aligned}
\]
\[
w(x) = \frac{q_o}{N} \left( \frac{L}{\pi} \right)^2 \sin \left( \frac{\pi x}{L} \right)
\]

\(a\) Calculate the load intensity

The cable deflects by \(w_o = 5m\) at the middle point \(x = L/2\)
\[
w\left( \frac{L}{2} \right) = w_o = \frac{q_o}{N} \left( \frac{L}{\pi} \right)^2 \sin \left( \frac{\pi L}{2L} \right) = \frac{q_o}{N} \left( \frac{L}{\pi} \right)^2
\]
\[
q_o = Nw_o \left( \frac{\pi}{L} \right)^2
\]

\(b\) Calculate the tension on the cable (N)

Using \(\frac{N}{EA} = \frac{1}{L} \int_0^L \left( \frac{d^2w}{dx^2} \right)^2 dx\) and \(w(x) = \frac{q_o}{N} \left( \frac{L}{\pi} \right)^2 \sin \left( \frac{\pi x}{L} \right)\)
\[
w'(x) = \frac{q_o}{N} \left( \frac{L}{\pi} \right)^2 \frac{\pi}{L} \cos \left( \frac{\pi x}{L} \right)
\]
\[
(w'(x))^2 = \left( \frac{q_o L}{N \pi} \right)^2 \cos^2 \left( \frac{\pi x}{L} \right)
\]
\[
N = \frac{EA}{2L} \int_0^L \left( \frac{q_o L}{N \pi} \right)^2 \cos^2 \left( \frac{\pi x}{L} \right) dx
\]
\[
= \frac{EAL}{2} \left( \frac{q_o}{N \pi} \right)^2 \left[ \frac{x}{2} + \frac{\sin \left( 2x \pi / L \right)}{4\pi / L} \right]_0^L
\]
\[
= \frac{EAL}{2} \left( \frac{q_o}{N \pi} \right)^2 \left( \frac{L}{2} + 0 \right)
\]
\[
N^3 = \frac{EA}{4} \left( \frac{q_o L}{\pi} \right)^2
\]
\[ N = \left( EA \left( \frac{q_o L}{2\pi} \right)^2 \right)^{\frac{1}{5}} \]

Given \( w_o = 5 \text{m} \), then \( q_o = Nw_o \left( \frac{\pi}{L} \right)^2 = 5N \left( \frac{\pi}{L} \right)^2 \)

\[ N^3 = \frac{EA}{4} \left( \frac{q_o L}{\pi} \right)^2 = \frac{EA}{4} \left( \frac{5N \left( \frac{\pi}{L} \right)^2 L}{\pi} \right)^2 = \frac{25EA}{4} \left( \frac{L}{\pi} \right)^2 N^2 \left( \frac{\pi}{L} \right)^4 \]

\[ N = \frac{25EA}{4} \left( \frac{\pi}{L} \right)^2 = \frac{25}{4} \left( 2.1 \times 10^3 \right) \times 10^6 \left[ \frac{\pi}{4} (60 \times 10^{-3})^2 \right] \left( \frac{\pi}{1 \times 10^3} \right)^2 \]

\[ N = 36626 \text{N} \]

c) Calculate the tension stress on the cable

\[ \sigma = \frac{N}{A} = \frac{36626}{\pi \left( 60 \times 10^{-3} \right)^2} = 12.95 \text{MPa} \]

\[ \sigma = 12.95 \text{MPa} \]

d) Compare with the yield stress and determine the safety factor

\[ \text{safety factor} = \frac{\text{yield stress}}{\text{working stress}} = \frac{300}{12.95} \]

\[ \text{safety factor} = 23.17 \]
Problem 6-3:
Plot the dimensionless deflections ($w_o/L$) versus the dimensionless line load for both bending and membrane (cable) solutions over a slender beam. At what dimensionless deflections will the bending and membrane solutions be equal, assuming a length to thickness ratio equal to 10?

Problem 6-3 Solution:
Recall bending and membrane solutions:

**Pure Bending**

$$w(x) = \frac{P_o}{EI} \left( \frac{\pi}{L} \right)^4 \sin \frac{\pi x}{L}$$

at $x = \frac{L}{2}$

$$w\left(\frac{L}{2}\right) = w_o = \frac{P_o}{EI} \left( \frac{\pi}{L} \right)^4$$

$$w_o = \frac{P_o}{EI} \left( \frac{L}{\pi} \right)^4$$

where $I = \frac{h^4}{12}$

$$w_o = \frac{P_o}{E \frac{h^4}{12}} \left( \frac{L}{\pi} \right)^4 = L \left( \frac{P_o L}{E h^2} \right) \left( \frac{12L^2}{\pi^4 h^2} \right)$$

$$\frac{w_o}{L} = \left( \frac{P_o L}{E h^2} \right) \left( \frac{12L^2}{\pi^4 h^2} \right)$$

**Membrane**

$$w(x) = \frac{P_o}{N} \left( \frac{\pi}{L} \right)^4 \sin \frac{\pi x}{L}$$

at $x = \frac{L}{2}$

$$w\left(\frac{L}{2}\right) = w_o = \frac{P_o}{N} \left( \frac{\pi}{L} \right)^4$$

$$w_o = \frac{P_o}{N} \left( \frac{L}{\pi} \right)^2$$

where $N = \left( EA \left( \frac{q_o L}{2\pi} \right) \right)^{\frac{1}{2}}$ (Problem 6-2)

so $N = \left( Eh^2 \left( \frac{q_o L}{2\pi} \right)^2 \right)^{\frac{1}{2}}$

$$w_o = \frac{P_o}{N} \left( \frac{L}{\pi} \right)^2 = \frac{P_o}{Eh^2} \left( \frac{1}{4} \left( \frac{2\pi}{q_o L} \right)^2 \right)^{\frac{1}{2}} \left( \frac{L}{\pi} \right)^2$$

$$= L \left( \frac{P_o L}{\pi^2} \right) \left( \frac{1}{E} \left( \frac{2\pi}{q_o L h} \right) \right)^{\frac{1}{2}} \left( \frac{L}{\pi} \right)^2$$

where $q_o = P_o$

rearrange the above expression

$$\frac{w_o}{L} = \left( \frac{P_o L}{E h^2} \right)^{\frac{1}{2}} \left( \frac{4}{\pi^4} \right)^{\frac{1}{2}}$$
Let’s call $\frac{w_o}{L} = y, \frac{P_o L}{E h^3} = x$

We want to plot

Bending $y = \left( \frac{12L^2}{\pi^3 h^2} \right) x$

Membrane $y = \left( \frac{4}{\pi^4} \right)^{1/3} x^{1/3}$

Use a length to thickness ratio equal to 10: $\frac{L}{h} = 10$

Bending $y = 12.32x$

Membrane $y = 0.345x^{1/3}$
At what dimensionless deflections will the bending and membrane solutions be equal?

\[
\frac{w_0}{L_{\text{bending}}} = \frac{w_0}{L_{\text{membrane}}}
\]

\[
12.32x = 0.345x^{\frac{1}{3}}
\]

\[
x = 0.005
\]

\[
\frac{w_0}{L} = 12.32 \times 0.005 = 0.0577
\]

So at \( \frac{P_a L}{Eh^2} = 0.005 \), the bending and membrane solutions will be equal, where the dimensionless deflections \( \frac{w_0}{L} = 0.0577 \).