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Introduction to Manufacturing Systems

Single-part-type, multiple stage systems

Stanley B. Gershwin
Laboratory for Manufacturing and Productivity
Massachusetts Institute of Technology
Flow Lines

... also known as a Production or Transfer Line.

- Machines are unreliable.
- Buffers are finite.
- In many cases, the operation times are constant and equal for all machines.
Flow Lines
Output Variability

Production output from a simulation of a transfer line.
If the machine is perfectly reliable, and its average operation time is $\tau$, then its maximum production rate is $\mu = 1/\tau$.

**Note:**

- Sometimes *cycle time* is used instead of *operation time*, but *BEWARE*: cycle time has two meanings!
- The other meaning is the time a part spends in a system. If the system is a single, reliable machine, the two meanings are the same.
Reliable Machines

Production rate in a two-machine reliable transfer line.

(Prime to be explained later.)
Inventory in a two-machine reliable transfer line.
Reliable Machines

Two Reliable Machines

Inventory in a two-machine reliable transfer line.

Single-part-type, multiple stage systems
Machine is either *up* or *down*.

MTTF = mean time to fail.

MTTR = mean time to repair

MTBF = MTTF + MTTR
Single Unreliable Machine

Production rate

- If the machine is unreliable, and
  - its average operation time is $\tau$,
  - its mean time to fail is MTTF,
  - its mean time to repair is MTTR,

then its maximum production rate is

$$\frac{1}{\tau} \left( \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}} \right)$$
Single Unreliable Machine

Proof

- Average production rate, while machine is up, is $1/\tau$.
- Average duration of an up period is MTTF.
- Average production during an up period is $MTTF/\tau$.
- Average duration of up-down period: $MTTF + MTTR$.
- Average production during up-down period: $MTTF/\tau$.
- Therefore, average production rate is $(MTTF/\tau)/(MTTF + MTTR)$.
Assumptions: Operation time is constant ($\tau$). Failure and repair times are geometrically distributed.

Let $p$ be the probability that a machine fails during any given operation. Then $p = \tau/\text{MTTF}$. 
Let \( r \) be the probability that \( M \) gets repaired during any operation time when it is down. Then
\[
 r = \frac{\tau}{MTTR}.
\]

Then the average production rate of \( M \) is
\[
\frac{1 - \frac{r}{\tau}}{r + p}.
\]

(Sometimes we forget to say “average.”)
So far, the machine really has three production rates:

- $1/\tau$ when it is up \textit{(short-term capacity)},

- 0 when it is down \textit{(short-term capacity)},

- $(1/\tau)(r/(r+p))$ on the average \textit{(long-term capacity)}.\"
Operation-Dependent Failures

- A machine can only fail while it is working — not idle.

- *(When buffers are finite, idleness also occurs due to blockage.)*

- **IMPORTANT!** MTTF *must* be measured in working time!

- This is the usual assumption.
Infinite-Buffer Lines

• **Starvation:** Machine $M_i$ is starved at time $t$ if Buffer $B_{i-1}$ is empty at time $t$.

**Assumptions:**

• A machine is not idle if it is not starved.

• The first machine is never starved.
The production rate of the line is the production rate of the *slowest* machine in the line — called the *bottleneck*.

- Slowest means *least average production rate*, where average production rate is calculated from one of the previous formulas.
Infinite-Buffer Lines

Bottleneck

• Production rate is therefore

\[ P = \min_i \frac{1}{\tau_i} \left( \frac{\text{MTTF}_i}{\text{MTTF}_i + \text{MTTR}_i} \right) \]

• and \( M_i \) is the bottleneck.
• The system is not in steady state.

• An increasing amount of inventory accumulates in the buffer upstream of the bottleneck.

• A finite amount of inventory appears downstream of the bottleneck.
Infinite-Buffer Line

Example 1

- Parameters:
  \( r_i = .1, p_i = .01, i = 1, \ldots, 9; r_{10} = .1, p_{10} = .03. \)

- Therefore, \( e_i = .909, i = 1, \ldots, 9; e_{10} = .769. \)
Infinite-Buffer Lines

Example 1

Single-part-type, multiple stage systems
Infinite-Buffer Lines

Example 1

- *Estimate the rate of growth of* $n_9(t)$, *the inventory in* $B_9$.  

*Single-part-type, multiple stage systems*
The second bottleneck is the slowest machine upstream of the bottleneck. An increasing amount of inventory accumulates just upstream of it.

A finite amount of inventory appears between the second bottleneck and the machine upstream of the first bottleneck.

A finite amount of inventory appears downstream of the first bottleneck.
Infinite-Buffer Lines

Example 2

- Parameters: $r_i = .1, p_i = .01, i = 1, \ldots, 4, 6, \ldots, 9$; $r_5 = .1, p_5 = .02, r_{10} = .1, p_{10} = .03$.

- Therefore, $e_i = .909, i = 1, \ldots, 4, 6, \ldots, 9$; $e_5 = .833, e_{10} = .769$. 
Infinite-Buffer Lines

Example 2

Single-part-type, multiple stage systems
Infinite-Buffer Lines

Example 2

- Estimate the rates of growth of $n_4(t)$ and $n_9(t)$. 

Single-part-type, multiple stage systems
Infinite-Buffer Lines

Example 2

• Note that when \( t \) is large enough, \( n_4(t) > n_9(t) \).

• Manufacturing people sometimes say that the easiest way to find the bottleneck of a line is to look for the greatest accumulation of inventory. Is that correct?
Infinite-Buffer Lines

Improvements

Questions:

• If we want to increase production rate, which machine should we improve?

• What would happen to production rate if we improved any other machine?
Simulation Note

- The simulations shown here were *time-based* rather than *event-based*.

- Time-based simulations are easier to program, but less general, less accurate, and slower, than event-based simulations.

- Primarily for systems where all event times are geometrically distributed.
Assume that some event occurs according to a geometric probability distribution and it has a mean time to occur of $T$ time steps. Then the probability that it occurs in any time step is $1/T$.

- At each time step, choose a $U[0,1]$ random number.
- If the number is less than or equal to $1/T$, the event has occurred. Change the state accordingly.
- If the number is greater than $1/T$, the event has not occurred. Change the state accordingly.
Zero-Buffer Lines

- If any one machine fails, or takes a very long time to do an operation, all the other machines must wait.

- Therefore the production rate is usually less — possibly much less — than the slowest machine.
Zero-Buffer Lines

- **Example:** Constant, unequal operation times, perfectly reliable machines.

  - The operation time of the line is equal to the operation time of the slowest machine, so the production rate of the line is equal to that of the slowest machine.
Zero-Buffer Lines

Constant, equal operation times, unreliable machines

Assumption: Failure and repair times are geometrically distributed.

Define $p_i = \tau / \text{MTTF}_i = \text{probability of failure during an operation}$.

Define $r_i = \tau / \text{MTTR}_i = \text{probability of repair during an interval of length } \tau \text{ when the machine is down}$. 

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Single-part-type, multiple stage systems
Buzacott’s Zero-Buffer Line Formula:

Let $k$ be the number of machines in the line. Then

$$P = \frac{1}{\tau} \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$
Zero-Buffer Lines

Production Rate

- Same as the earlier formula (Slides 9 and 12) when \( k = 1 \). The isolated production rate of a single machine \( M_i \) is

\[
\frac{1}{\tau} \left( \frac{1}{1 + \frac{p_i}{r_i}} \right) = \frac{1}{\tau} \left( \frac{r_i}{r_i + p_i} \right).
\]
Zero-Buffer Lines

Proof of formula

- Let $\tau$ (the operation time) be the time unit.
- *Approximation:* At most, one machine can be down.
- Consider a long time interval of length $T\tau$ during which Machine $M_i$ fails $m_i$ times ($i = 1, \ldots, k$).
- Without failures, the line would produce $T$ parts.

<table>
<thead>
<tr>
<th>$M_3$</th>
<th>$M_5$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_1$</th>
<th>$M_4$</th>
</tr>
</thead>
</table>

All up  Some machine down

Single-part-type, multiple stage systems
Zero-Buffer Lines

Proof of formula

- The average repair time of $M_i$ is $\tau/r_i$ each time it fails, so the total system down time is close to

$$D\tau = \sum_{i=1}^{k} \frac{m_i\tau}{r_i}$$

where $D$ is the number of operation times in which a machine is down.
• The total up time is approximately

\[ U_{\tau} = T_{\tau} - \sum_{i=1}^{k} \frac{m_i \tau}{r_i} \]

• where \( U \) is the number of operation times in which all machines are up.
Zero-Buffer Lines

Proof of formula

- Since the system produces one part per time unit while it is working, it produces $U$ parts during the interval of length $T\tau$.
- Note that, approximately,

$$m_i = p_i U$$

because $M_i$ can only fail while it is operational.
Zero-Buffer Lines

Proof of formula

- Thus,

\[ U_T = T_T - U_T \sum_{i=1}^{k} \frac{p_i}{r_i}, \]

or,

\[ \frac{U}{T} = E_{ODF} = \frac{1}{k} \left( 1 + \sum_{i=1}^{k} \frac{p_i}{r_i} \right) \]
Zero-Buffer Lines

\( p_i \) and \( r_i \) and \( p_i/r_i \)

and

\[
P = \frac{1}{\tau} \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}
\]

- Note that \( P \) is a function of the ratio \( p_i/r_i \) and not \( p_i \) or \( r_i \) separately.
- The same statement is true for the infinite-buffer line.
- However, the same statement is not true for a line with finite, non-zero buffers.

Single-part-type, multiple stage systems
Zero-Buffer Lines

Improvements

Questions:

- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?
Zero-Buffer Lines

$P$ as a function of $p_i$

All machines are the same except $M_i$. As $p_i$ increases, the production rate decreases.
Zero-Buffer Lines

$P$ as a function of $p_i$

All machines are the same. As the line gets longer, the production rate decreases.
Finite-Buffer Lines

Motivation for buffers: recapture some of the lost production rate.

Cost

- in-process inventory/lead time
- floor space
- material handling mechanism

Single-part-type, multiple stage systems
Finite-Buffer Lines

- **Infinite buffers**: delayed downstream propagation of disruptions (*starvation*) and *no* upstream propagation.

- **Zero buffers**: instantaneous propagation in both directions.

- **Finite buffers**: delayed propagation in both directions.

  ★ **New phenomenon**: *blockage*.

- **Blockage**: Machine $M_i$ is blocked at time $t$ if Buffer $B_i$ is full at time $t$. 

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*Single-part-type, multiple stage systems*
Finite-Buffer Lines

- **Difficulty:**
  - No simple formula for calculating production rate or inventory levels.

- **Solution:**
  - Simulation
  - Analytical approximation
  - *Exact analytical solution for two-machine lines only.*
Two Machine, Finite-Buffer Lines
Markov Chain Model

- Exact solution is available to Markov process model of a two-machine line.
- \textit{Discrete time-discrete state Markov process:}

$$\text{prob}\{X(t + 1) = x(t + 1)|X(t) = x(t),
X(t - 1) = x(t - 1), X(t - 2) = x(t - 2), \ldots\} =$$

$$\text{prob}\{X(t + 1) = x(t + 1)|X(t) = x(t)\}$$
State Space

Here, \( X(t) = (n(t), \alpha_1(t), \alpha_2(t)) \), where

- \( n \) is the number of parts in the buffer; \( n = 0, 1, \ldots, N \).
- \( \alpha_i \) is the repair state of \( M_i; \ i = 1, 2 \).

\* \( \alpha_i = 1 \) means the machine is up or operational;
\* \( \alpha_i = 0 \) means the machine is down or under repair.
Two Machine, Finite-Buffer Lines

Simulations

\[ r_1 = 0.1, p_1 = 0.01, r_2 = 0.1, p_2 = 0.01, N = 10 \]
Two Machine, Finite-Buffer Lines Simulations

\[ r_1 = .1, p_1 = .01, r_2 = .1, p_2 = .01, N = 100 \]
Two Machine, Finite-Buffer Lines Simulations

\[ r_i = .1, \ i = 1, 2, \ p_1 = .02, \ p_2 = .01, \ N = 100 \]
Two Machine, Finite-Buffer Lines
Simulations

\[ r_i = .1, i = 1, 2, p_1 = .01, p_2 = .02, N = 100 \]
Several models available:

- **Deterministic processing time**, or **Buzacott model**: deterministic processing time, geometric failure and repair times; discrete state, discrete time.
Two Machine, Finite-Buffer Lines

State Transition Graph for Deterministic Processing Time, Two-Machine Line

Key
- States:
  - Transient
  - Non-transient
  - Boundary
  - Internal

Transitions:
- Out of transient states
- Out of non-transient states
- To increasing buffer level
- To decreasing buffer level
- Unchanging buffer level

Single-part-type, multiple stage systems

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Two Machine, Finite-Buffer Lines

- **Exponential processing time:** exponential processing, failure, and repair time; discrete state, continuous time.

- **Continuous material, or fluid:** deterministic processing, exponential failure and repair time; mixed state, continuous time.

*Single-part-type, multiple stage systems*
Two Machine, Finite-Buffer Lines

$\tau = 1.$
$p_1 = 0.01$
$r_2 = 0.1$
$p_2 = 0.01$

$P$ vs. $N$ for different $r_1$ values:
- $r_1 = 0.06$
- $r_1 = 0.08$
- $r_1 = 0.1$
- $r_1 = 0.12$
- $r_1 = 0.14$

Single-part-type, multiple stage systems
Two Machine, Finite-Buffer Lines

Discussion:

- What is $P$ when $N = 0$?
- Why are the curves increasing?
- Why do they reach an asymptote?
- What is the limit of $P$ as $N \to \infty$?
- Why are the curves with smaller $r_1$ lower?
Two Machine, Finite-Buffer Lines

Discussion:

• Why are the curves increasing?
• Why different asymptotes?
• What is \( \bar{n} \) when \( N = 0 \)?
• What is the limit of \( \bar{n} \) as \( N \to \infty \)?
• Why are the curves with smaller \( r_1 \) lower?
Two Machine, Finite-Buffer Lines

Problem: Select $M_1$ and $N$ to maximize profit (revenue-[capital cost+inventory cost])

- What can you say (qualitatively) about the optimal buffer size for a given $M_1$?

- How should it be related to $r_i, p_i$?
Two Machine, Finite-Buffer Lines

Problem: Select $M_1$ and $N$ so that $P = .88$ and profit is maximized

- Observation: If $M_1$ is better, $N$ can be smaller.
Two Machine, Finite-Buffer Lines

Should we prefer short, frequent, disruptions or long, infrequent, disruptions?

- $r_2 = 0.8$, $p_2 = 0.09$, $N = 10$
- $r_1$ and $p_1$ vary together and $\frac{r_1}{r_1+p_1} = .9$
- **Answer:** evidently, short, frequent failures.
- **Why?**
Two Machine, Finite-Buffer Lines

 Improvements

Questions:

• If we want to increase production rate, which machine should we improve?

• What would happen to production rate if we improved any other machine?
Two Machine, Finite-Buffer Lines

Improvements to non-bottleneck machine.

Single-part-type, multiple stage systems
Two Machine, Finite-Buffer Lines

- Inventory increases as the (non-bottleneck) upstream machine is improved and as the buffer space is increased.
- If the downstream machine were improved, the inventory would be less and it would increase much less as the space increases.

Single-part-type, multiple stage systems
Two Machine, Finite-Buffer Lines

*Exponential* — discrete material, continuous time

- $\mu_i \delta t =$ the probability that $M_i$ completes an operation in $(t, t + \delta t)$;

- $p_i \delta t =$ the probability that $M_i$ fails during an operation in $(t, t + \delta t)$;

- $r_i \delta t =$ the probability that $M_i$ is repaired, while it is down, in $(t, t + \delta t)$;
Two Machine, Finite-Buffer Lines

Continuous — continuous material, continuous time

- \( \mu_i \delta t = \) the amount of material that \( M_i \) processes, while it is up, in \( (t, t + \delta t) \);

- \( p_i \delta t = \) the probability that \( M_i \) fails, while it is up, in \( (t, t + \delta t) \);

- \( r_i \delta t = \) the probability that \( M_i \) is repaired, while it is down, in \( (t, t + \delta t) \);

Single-part-type, multiple stage systems

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Two Machine, Finite-Buffer Lines

- $r_1 = 0.09$, $p_1 = 0.01$, $\mu_1 = 1.1$
- $r_2 = 0.08$, $p_2 = 0.009$
- $N = 20$
- *Explain the shapes of the graphs.*
Two Machine, Finite-Buffer Lines

- Explain the shapes of the graphs.
No-variability limit: a continuous model where both machines are reliable, and processing rate \( \mu_i' \) of machine \( i \) in the no-variability is the same as the isolated production rate of machine \( i \) in the other cases. That is, \( \mu_i' = \mu_i r_i / (r_i + p_i) \).
Long Lines

• Difficulty:
  ★ No simple formula for calculating production rate or inventory levels.
  ★ State space is too large for exact numerical solution.

▶ If all buffer sizes are $N$ and the length of the line is $k$, the number of states is $S = 2^k (N + 1)^{k-1}$.

▶ if $N = 10$ and $k = 20$, $S = 6.41 \times 10^{25}$.

★ Decomposition seems to work successfully.
Decomposition

- Decomposition breaks up systems and then reunites them.
- Conceptually: put an observer in a buffer, and tell him that he is in the buffer of a two-machine line.
- Question: What would the observer see, and how can he be convinced he is in a two-machine line? Construct the two-machine line. Construct all the two-machine lines.
Decomposition

- Consider an observer in Buffer $B_i$.
  - Imagine the material flow process that the observer sees *entering* and the material flow process that the observer sees *leaving* the buffer.

- We construct a two-machine line $L(i)$
  - ie, we find machines $M_u(i)$ and $M_d(i)$ with parameters $r_u(i)$, $p_u(i)$, $r_d(i)$, $p_d(i)$, and $N(i) = N_i$)

such that an observer in its buffer will see almost the same processes.

- The parameters are chosen as functions of the behaviors of the other two-machine lines.
Decomposition

\[ M_{i-2} \rightarrow B_{i-2} \rightarrow M_{i-1} \rightarrow B_{i-1} \rightarrow M_i \rightarrow B_i \rightarrow M_{i+1} \rightarrow B_{i+1} \rightarrow M_{i+2} \rightarrow B_{i+2} \rightarrow M_{i+3} \]

Line \( L(i) \)

\[ M_u(i) \rightarrow M_d(i) \]

Single-part-type, multiple stage systems
Decomposition

There are $4(k - 1)$ unknowns. Therefore, we need

- $4(k - 1)$ equations, and

- an algorithm for solving those equations.
Decomposition Equations

- *Conservation of flow*, equating all production rates.
- *Flow rate/idle time*, relating production rate to probabilities of starvation and blockage.
- *Resumption of flow*, relating $r_u(i)$ to upstream events and $r_d(i)$ to downstream events.
- *Boundary conditions*, for parameters of $M_u(1)$ and $M_d(k-1)$. 
Decomposition

Equations

• This is a set of $4(k - 1)$ equations.

• All the quantities in these equations are
  
  ★ specified parameters, or
  
  ★ unknowns, or
  
  ★ functions of parameters or unknowns derived from the two-machine line analysis.
Decomposition
Algorithm

**DDX algorithm** : due to Dallery, David, and Xie (1988).

1. Guess the downstream parameters of $L(1) \ (r_d(1), p_d(1))$. Set $i = 2$.

2. Use the equations to obtain the upstream parameters of $L(i) \ (r_u(i), p_u(i))$. Increment $i$.

3. Continue in this way until $L(k - 1)$. Set $i = k - 2$.

4. Use the equations to obtain the downstream parameters of $L(i)$. Decrement $i$.

5. Continue in this way until $L(1)$.

6. Go to Step 2 or terminate.
Examples

Three-machine line – production rate.

\[ r_1 = r_2 = r_3 = 0.2 \]

\[ p_1 = 0.05 \]

\[ N_1 = N_2 = 5 \]
Examples

Three-machine line – total average inventory

\[ p_2 = 0.05 \]
\[ p_3 = 0.1 \ 0.2 \ 0.3 \ 0.4 \ 0.5 \ 0.6 \ 0.7 \]

\[ r_1 = r_2 = r_3 = 0.2 \]
\[ p_1 = 0.05 \]
\[ N_1 = N_2 = 5 \]
Examples

Distribution of material in a line with identical machines and buffers. *Explain the shape.*

*Single-part-type, multiple stage systems*
Effect of a bottleneck. Identical machines and buffers, except for $M_{10}$.
Examples

Continuous material model.

- Eight-machine, seven-buffer line.
- For each machine, $r = 0.075, \ p = 0.009, \ \mu = 1.2$.
- For each buffer (except Buffer 6), $N = 30$.

*Single-part-type, multiple stage systems*
Which $\bar{n}_i$ are decreasing and which are increasing?

Why?
Examples

Which has a higher production rate?

• 9-Machine line with two buffering options:
  ✤ 8 buffers equally sized; or

  ![Diagram 1](image1)

  ✤ 2 buffers equally sized.

  ![Diagram 2](image2)
Examples

- Continuous model; all machines have $r = 0.019$, $p = 0.001$, $\mu = 1$.
- What are the asymptotes?
- Is 8 buffers always faster?

**Total Buffer Space**

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Optimal buffer space distribution

- Design the buffers for a 20-machine production line.
- The machines have been selected, and the only decision remaining is the amount of space to allocate for in-process inventory.

  The goal is to determine the smallest amount of in-process inventory space so that the line meets a production rate target.
Optimal buffer space distribution

• The common operation time is one operation per minute.

• The target production rate is .88 parts per minute.
Optimal buffer space distribution

- **Case 1**  MTTF = 200 minutes and MTTR = 10.5 minutes for all machines ($P = .95$ parts per minute).

- **Case 2**  Like Case 1 except Machine 5. For Machine 5, MTTF = 100 and MTTR = 10.5 minutes ($P = .905$ parts per minute).

- **Case 3**  Like Case 1 except Machine 5. For Machine 5, MTTF = 200 and MTTR = 21 minutes ($P = .905$ parts per minute).
Optimal buffer space distribution

Are buffers really needed?

<table>
<thead>
<tr>
<th>Line</th>
<th>Production rate with no buffers, parts per minute</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>.487</td>
</tr>
<tr>
<td>Case 2</td>
<td>.475</td>
</tr>
<tr>
<td>Case 3</td>
<td>.475</td>
</tr>
</tbody>
</table>

Yes.

How were these numbers calculated?

Single-part-type, multiple stage systems
Optimal buffer space distribution

Solution

<table>
<thead>
<tr>
<th>Line</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>430</td>
</tr>
<tr>
<td>Case 2</td>
<td>485</td>
</tr>
<tr>
<td>Case 3</td>
<td>523</td>
</tr>
</tbody>
</table>
Optimal buffer space distribution

- Observation from studying buffer space allocation problems:
  - Buffer space is needed most where buffer level variability is greatest!
Profit as a function of buffer sizes

- Three-machine, continuous material line.
- $r_i = .1$, $p_i = .01$, $\mu_i = 1$.
- $\Pi = 1000P(N_1, N_2) - (\bar{n}_1 + \bar{n}_2)$.

Single-part-type, multiple stage systems
Assembly

- Decomposition can be extended to assembly systems.
- Propagation of disturbances is more complex:

![Diagram of assembly process]

*Single-part-type, multiple stage systems*
Assembly

Question: How should an assembly system be structured?

- Add parts to a growing assembly or form subassemblies and then assemble them?
- Production rates are roughly the same, but inventories can be affected.
Assembly

Single-part-type, multiple stage systems
Assembly

High buffer utilization

Low buffer utilization

Effective bottleneck

Bottleneck

Effective bottleneck

Single-part-type, multiple stage systems
Assembly

Bottleneck moved from head of segment to tail

Single-part-type, multiple stage systems
Assembly

Effective bottleneck

Single-part-type, multiple stage systems
Assembly

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