Announcements

- Recommended Reading: Griffiths, sections 1.1, 1.2, 1.4 and 1.5.

Problem Set 3

1. **Exercises with commutators** [10 points] Let $A, B,$ and $C$ be linear operators.


   (c) Show that $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

   (d) Calculate $[\hat{x}^n, \hat{p}]$ and $[x, \hat{p}^n]$ for $n$ an arbitrary integer greater than zero.

   (e) Calculate $[\hat{x}^{\hat{p}^2}, \hat{x}\hat{p}]$ and $[\hat{x}\hat{p}, \hat{p}^2]$.

2. **Simple tests of the stationary phase approximation** [10 points]

   In here we consider integrals of the form
   \[
   \Psi(x) = \int_{-\infty}^{\infty} dk \Phi(k)e^{ikx},
   \]
   where $\Phi(k)$ is a function that is sharply localized around $k = k_0$. In each of the following cases use the stationary phase argument to predict the location of the peak of $|\Psi(x)|$. Then compute the integral exactly to find $\Psi(x)$, $|\Psi(x)|$, and to confirm your prediction.

   (a) $\Phi(k) = e^{-L^2(k-k_0)^2}$, where $L$ is a constant with units of length.

   (b) $\Phi(k) = e^{-L^2(k-k_0)^2}e^{-ikx_0}$, where $x_0$ and $L$ are constants with units of length.

   **Useful integral:** Valid for complex constants $a$ and $b$, with real part of $a$ positive:
   \[
   \int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \quad \text{when Re}(a) > 0.
   \]
3. **Galilean invariance of the free Schrödinger equation.** [15 points]

Show that the free-particle one-dimensional Schrödinger equation for the wavefunction $\Psi(x, t)$:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\hbar^2 \frac{\partial^2 \Psi}{2m \partial x^2},$$

is invariant under Galilean transformations

$$x' = x - vt, \quad t' = t.$$

By this we mean that there is a $\Psi'(x', t')$ of the form

$$\Psi'(x', t') = f(x, t) \Psi(x, t),$$

where the function $f(x, t)$ involves $x, t, h, m$ and $v$, and such that $\Psi'$ satisfies the corresponding Schrödinger equation in primed variables.

$$i\hbar \frac{\partial \Psi'}{\partial t'} = -\hbar^2 \frac{\partial^2 \Psi'}{2m \partial x'^2}.$$

(a) Find the function $f(x, t)$. [Hint: Note that the function $f(x, t)$ cannot depend on any observable of $\Psi$; it is a universal function that is used to transform any $\Psi$. Thus if $\Psi$ is a (single) plane wave, $f$ cannot depend on its momentum or its energy.]

(b) Demonstrate that the plane wave solution

$$\Psi(x, t) = A e^{i(kx - \omega t)}$$

transforms as expected. In other words, give $\Psi'$ and show that it represents, in the primed reference frame, a particle with the expected momentum and energy.

4. **Re-do current conservation in 3D** [10 points]

In class we derived the expression for the one-dimensional probability current $J(x, t)$ starting from $\rho(x, t) = |\Psi(x, t)|^2$ and using the one-dimensional Schrödinger equation to write

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0.$$

Repeat the same steps starting from

$$\rho(x, t) = |\Psi(x, t)|^2,$$

and using the three-dimensional Schrödinger equation to derive the form of the probability current $\mathbf{J}(x, t)$ that should appear in the conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$
5. **Time evolution of an overlap between two states.** [10 points] (Merzbacher)

Consider a wavefunction that at time $t = 0$ is the superposition of two widely separated narrow wave packets $\Psi_1$ and $\Psi_2$:

$$\Psi(x, 0) = \Psi_1(x, 0) + \Psi_2(x, 0).$$

Each packet is separately normalizable. We define the overlap integral $\gamma(t)$ as

$$\gamma(t) \equiv \int_{-\infty}^{\infty} \Psi_1^\dagger(x, t)\Psi_2(x, t)dx.$$

At time equal to zero the value of $|\gamma(0)|$ is very small. As the packets evolve and spread, what will happen to the value of $|\gamma(t)|$? Will it increase as the packets overlap?

6. **Probability current in one dimension** [10 points]

Calculate the probability current $J(x)$ for the following wavefunctions, all of which refer to $t = 0$:

(a) $\Psi(x) = A e^{\gamma x}$. Here $A$ is a complex constant and $\gamma$ is a real constant.

(b) $\Psi(x) = N(x)e^{iS(x)/\hbar}$. Here $N(x)$ and $S(x)$ are real.

(c) $\Psi(x) = Ae^{ikx} + Be^{-ikx}$. Here $A, B$ are complex constants and $k$ is real.