Quantum Field Theory II (8.324) Fall 2010
Assignment 7

Readings

- Peskin & Schroeder chapters 10, 12, 13.
- Weinberg vol 1 chapter 12 and Vol 2 chapter 18.

Problem Set 7

1. Renormalization group properties (30 points)

   (a) Consider a coupling constant $\lambda$ and a redefined coupling constant $\bar{\lambda}(\lambda)$. Find the general transformation law for the beta function, namely the relation between $\beta(\lambda)$ and $\beta(\bar{\lambda})$. If we think of $\lambda$ as a coordinate we see that $\beta$ transforms as a tensor. What kind of tensor?

   (b) Assume that
   
   $\beta(\lambda) = b_2 \lambda^2 + b_3 \lambda^3 + b_4 \lambda^4 + \cdots$
   
   and consider the perturbatively defined and invertible coupling constant redefinition:
   
   $\bar{\lambda}(\lambda) = \lambda + a_2 \lambda^2 + a_3 \lambda^3 + \cdots$.

   Calculate $\bar{\beta}(\bar{\lambda})$ writing it in the form
   
   $\bar{\beta}(\bar{\lambda}) = \bar{b}_2 \bar{\lambda}^2 + \bar{b}_3 \bar{\lambda}^3 + \bar{b}_4 \bar{\lambda}^4 + \cdots$

   Verify that:
   
   i. $\bar{b}_2 = b_2$ and $\bar{b}_3 = b_3$.
   
   ii. It is possible to make $\bar{b}_4$ anything you want by such a coupling redefinition.

   iii. Let $\lambda = \lambda_F$ denote a fixed point. Show that $\bar{\lambda} = \bar{\lambda}_F$ is also a fixed point. How are the derivatives $\beta'$ and $\bar{\beta}'$ related at the fixed point?
(c) Consider the differential equation for a massless coupling $g$

$$\mu \frac{dg}{d\mu} = -bg^2 - cg^3 - dg^4 - \cdots$$

(1)

with $b, c, d$ numerical constants. Show that one can write a solution to the above equation in the form

$$\ln \frac{\mu}{\Lambda} = \frac{1}{bg(\mu)} + \frac{c}{b^2} \ln bg(\mu) + \mathcal{O}(g(\mu))$$

(2)

where $\Lambda$ is an integration constant, which can be considered as a dynamically generated scale. Argue that $\Lambda$ is renormalization group invariant.

(d) More generally, show that in a renormalizable theory with a dimensionless coupling constant $g(\mu)$ and no other dimensional parameter (like in a non-Abelian gauge theory), dimensional transmutation happens. That is, show that $g(\mu)$ can be written in a form

$$g(\mu) = f \left( \log \left( \frac{\mu}{\Lambda} \right) \right)$$

(3)

with $\Lambda$ a universal scale and $f$ a function depending on the specific theory. (Inverting (2) gives a specific example of $f$.)

2. **Asymptotic freedom in six-dimensional field theory (30 points)**

Consider the field theory

$$\mathcal{L} = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{g_0}{6} \phi^3$$

(4)

Calculate the $\beta$-function associated with the coupling $g_0$. Use dimensional regularization and minimal subtraction. You will find that this is a theory where the coupling constant becomes weaker at higher energies i.e. it is an asymptotically free theory.

3. **Asymptotic symmetry (40 points)**

Peskin & Schroeder prob. 12.3. (In intermediate steps you can use results of section 10.2 of Peskin & Schroeder.)