8.325 Homework 2
Iain Stewart, Feb. 23, 2007
Due: In lecture March 8.

Problem 1) MS beta-function with multiple dimensionful couplings

Consider a field theory with a set of couplings $g_1$, $g_2$, ..., $g_t$, ... which we'll call $\tilde{g}$. Your goal is to derive an expression for the beta-functions in the MS-scheme that is valid at any order in perturbation theory. (It will also be valid for both renormalizable theories and “non-renormalizable” theories with irrelevant operators.) Let $\Delta_t(d) = \Delta_t + \epsilon \rho_t$ be the dimension of the bare coupling $g_t^\text{bare}$ in $d = 4 - 2\epsilon$ dimensions. We can define a dimensionless running coupling in the MS-scheme by

$$g_t^\text{bare} \mu^{-\Delta_t(d)} = g_t(\mu, d)Z_{g_t}(\tilde{g}),$$

where $g_t(\mu, d)$ is analytic in $d$ and $Z_{g_t}$ is a series in $1/\epsilon$. Prove that the beta-function for $g_t \equiv g_t(\mu, d)$ is

$$\beta(g_t, d) = \mu\frac{dg_t}{d\mu}g_t = -\epsilon \rho_t g_t - \Delta_t g_t + g_t \sum_m \frac{a_t^m(\tilde{g})}{dg_t} \rho_m g_m,$$

where $a_t^m$ is the coefficient of the $1/\epsilon$ pole term in $Z_{g_t}$. Find a recursion relation for the coefficients of the higher poles, $a_k(\tilde{g})$. How are these results modified in the \MS\-scheme? [Side Remark: in this notation the standard beta-function is $\beta(g_t) = \beta(g_t, 4)$.

Problem 2) Scheme and gauge dependence of beta-functions

Consider a renormalizable non-abelian gauge theory. Let $g(\mu_R)$ and $\beta(g)$ be the renormalized coupling and beta-function in a mass-independent renormalization scheme. This could be the MS-scheme or an offshell momentum subtraction scheme, etc.

a) Prove that in any scheme the first term in the series expansion of $\beta(g)$ is gauge independent.

b) Prove that the first two terms in the series expansion of $\beta(g)$ are scheme independent.

c) Finally, prove that in the MS-scheme the renormalized coupling is gauge independent. Thus in this scheme all terms in $\beta(g)$ are gauge independent. Use this result to strengthen the statement in a).

Problem 3) QCD beta-function in background field gauge

Using background field gauge derive the lowest order beta-function for QCD in a massless scheme by carrying out the computation of the 3 diagrams discussed in lecture. Express your result in terms of the quadratic adjoint Casmir $C_A$ and the number of quark flavors $n_f$. What is the beta-function for an SU(2) gauge theory with 6 flavors?
Problem 4) The QCD running coupling and thresholds

We originally motivated the discussion of mass-independent renormalization schemes by considering $\mu \gg m$. However, these schemes are well defined regardless of the relation between $\mu$ and $m$, and so we can consider using them for $\mu \simeq m$ and for that matter $\mu \leq m$. In this problem we’ll explore how this works in QCD at one-loop order.

In lecture you saw that in a mass-dependent scheme for QED the electron “decouples” from the beta-function as we go below its mass scale, falling off quite rapidly, $\beta(\mu \ll m) \sim \mu^2/m^2$. However, if you consider the mass-independent QED beta-function with an $e^-$, $\mu^-$, and $\tau^-$ then apriori you have the same beta-function for $m_e \ll \mu \ll m_\mu$, $\mu \gg m_\tau$, or any other value of $\mu$. The issue with a mass-independent scheme is that $\alpha(\mu)$ is not smart enough to know that heavy particles in the field theory should decouple. This is an important piece of physics that we’re going to build into the mass-independent schemes by hand. To do this consider evolving the coupling down from a $\mu \gg m_\tau$ with the beta-function with 3-leptons, $n_\ell = 3$. When we reach $\mu = m_\tau$ we’ll decree that the tau is removed from our theory, so that below this scale we switch to using a beta function with $n_\ell = 2$. Let’s call the coupling in the theory with $n_\ell$ leptons $\alpha^{(n_\ell)}(\mu)$. To ensure that this process does not disturb our field theory too much, we’ll demand that scattering amplitudes computed in the theory with $n_\ell = 3$ and $n_\ell = 2$ are the same at $\mu = m_\tau$. At lowest order in perturbation theory this just implies continuity of the coupling at the boundary, $\alpha^{(3)}(m_\tau) = \alpha^{(2)}(m_\tau)$. At each mass-threshold we’ll repeat the above procedure to build in the decoupling by hand.\(^1\)

Let’s apply the same logic to QCD to give couplings $\alpha^{(n_f)}_s(\mu)$ which satisfy the mass-independent beta-function equation for $n_f$-flavors. From $\alpha^{(n_f)}_s(\mu)$ we can define the integration constant $\Lambda^{(n_f)}_{QCD}$.

\begin{itemize}
  \item[a)] Let $\alpha^{(5)}_s(m_Z) = 0.118$ with the physical $Z$-boson mass be your initial condition. Let’s take the mass of the bottom and charm quarks to be $m_b = 5\text{ GeV}$ and $m_c = 2\text{ GeV}$ (slightly heavier than in nature). Using the decoupling procedure described above, compute $\alpha^{(3)}(\mu = 1.5\text{ GeV})$. Compare it numerically to $\alpha^{(5)}_s(\mu = 1.5\text{ GeV})$.

  \item[b)] Consider QCD with $m_u = m_d = m_s = 0$ and note that in this theory the proton-mass is dominated by non-perturbative dynamics of these three quarks. Hence we expect the proton mass $m_p \propto \Lambda^{(3)}_{QCD}$. Derive a relation between $\Lambda^{(6)}_{QCD}$ and $\Lambda^{(3)}_{QCD}$ that only involves heavy-quark masses. Now imagine that the strong coupling is fixed at some very high scale (eg. at a unification scale $\sim 10^{16}\text{ GeV}$), and predict how much the proton-mass changes if you double the b-quark mass.
\end{itemize}

\(^1\)In the context of effective field theory this procedure of removing particles is known as “integrating out” a massive degree of freedom, and ensuring the continuity of the S-matrix elements is known as “matching”.