Ito integral for simple processes

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1 Ito integral for simple processes. Ito isometry

Consider a Brownian motion $B_t$ adopted to some filtration $\mathcal{F}_t$ such that $(B_t, \mathcal{F}_t)$ is a strong Markov process. As an example we can take filtration generated by the Brownian motion itself. Our goal is to give meaning to expressions of the form $\int X_t dB_t = \int X_t(\omega) dB_t(\omega)$, where $X_t$ is some stochastic process which is adapted to the same filtration as $B_t$. We will primarily deal with the case $X_t \in L^2$, although it is possible to extend definitions to more general processes using the notion of local martingales. As in the case of usual integration, the idea is to define $X_t(\omega) dB_t(\omega)$ as some kind of a limit of (random) sums $\sum_j X_{t_j}(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))$ and show that the limit exists in some appropriate sense. As $X_t$ we can take all kinds of processes, including $B_t$ itself. For example we will show that $\int_0^T B_t dB_t$ makes sense and equals $(1/2)B_T^2 - (1/2)T$.

Definition 1. A process $X \in L_2 = L_2(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called simple if there exists a countable partition $\Pi : 0 = t_0 < \cdots < t_n < \cdots$ with $\lim_n t_n = \infty$ such that $X_t(\omega) = X_{t_j}(\omega)$ for all $t \in [t_j, t_{j+1}), j = 0, 1, 2, \ldots$ for all $\omega \in \Omega$. The subspace of simple processes is denoted by $L^0_2$.

We assume that partition is such that $t_j \to \infty$ as $j \to \infty$. It is important to note that we assume that the partition $\Pi$ does not depend on $\omega$. Thus not every piece-wise constant process is a simple process. Give an example of a piece-wise constant process which is not simple. Note that since $X_t \in \mathcal{F}_t$ we have $X_{t_j} \in \mathcal{F}_{t_j}$ for each $j$. As an example of simple process, fix any partition $\Pi$ and a process $X_t \in L_2$ and consider the process $\hat{X}_t(\omega)$ defined by $\hat{X}_t(\omega) = \ldots$
$X_{t_j}(\omega)$, where $t_j$ is defined by $t \in [t_j, t_{j+1})$. In the definition it is important that $\tilde{X}_t = X_{t_j}$ and not $X_{t_{j+1}}$. Observe that the latter is not necessarily adopted to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$.

Given a simple process $X$ and $t$, define its integral by

$$I_t(X(\omega)) = \sum_{0 \leq j \leq n-1} X_{t_j}(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)) + X_{t_n}(\omega)(B_t(\omega) - B_{t_n}(\omega)),$$

where $n = \max\{j : t_j \leq t\}$. Observe that $I_t(X)$ is an a.s. continuous function (as $B_t$ is a.s. continuous).

**Theorem 1.** The following properties hold for $I_t(X)$

1. $I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y)$.
2. $\mathbb{E}[I_t^2(X)] = \mathbb{E}\left[ \int_0^t X_s^2 \, ds \right]$ [Ito isometry].
3. $I_t(X) \in \mathcal{M}_{2,c}$,
4. $\mathbb{E}[(I_t(X) - I_s(X))^2 | \mathcal{F}_s] = \mathbb{E}\left[ \int_s^t X_u^2 \, du \right], \forall 0 \leq s < t \leq T.$

Notice that (4) is a generalization of Ito isometry. We only prove Ito isometry, the proof of (4) follows along the same lines.

**Proof.** Define $t_n = t$ for convenience. We begin with (1). Let $\{t^1_j\}$ and $\{t^2_j\}$ be partitions corresponding to simple processes $X$ and $Y$. Consider a partition $\{t_j\}$ obtained as a union of these two partitions. For each $t_j$ which belongs to the second partition but not the first define $X_{t_j} = X_{t^1_j}$, where $t^1_j$ is the largest point not exceeding $t_j$. Do a similar thing for $Y$. Observe that now $X_t = X_{t_j}$ for $t \in [t_j, t_{j+1})$. The linearity of Ito integral then follows straight from the definition.

Now for (2) we have

$$\mathbb{E}[I_t^2(X)] = \sum_{0 \leq j_1, j_2 \leq n-1} \mathbb{E}[X_{t_{j_1}} X_{t_{j_2}} (B_{t_{j_1+1}} - B_{t_{j_1}})(B_{t_{j_2+1}} - B_{t_{j_2}})].$$

When $j_1 < j_2$ we have

$$\mathbb{E}[X_{t_{j_1}} X_{t_{j_2}} (B_{t_{j_1+1}} - B_{t_{j_1}})(B_{t_{j_2+1}} - B_{t_{j_2}})] = 0$$

which we obtain by conditioning on $\mathcal{F}_{t_{j_2}}$, using the tower property and observing that all of the random variables involved except for $B_{t_{j_2+1}}$ are measurable with respect to $\mathcal{F}_{t_{j_2}}$ (recall that $\mathcal{F}_{t_{j_1}} \subset \mathcal{F}_{t_{j_2}}$).
Now when \( j_1 = j_2 = j \) we have
\[
\mathbb{E}[X^2_{t_j}(B_{t_{j+1}} - B_{t_j})^2] = \mathbb{E}[X^2_{t_j}((B_{t_{j+1}} - B_{t_j})^2|F_{t_j})] \\
= \mathbb{E}[X^2_{t_j}(t_{j+1} - t_j)].
\]
Combining, we obtain
\[
\mathbb{E}[I_t^2(X)] = \sum_j \mathbb{E}[X^2_{t_j}(t_{j+1} - t_j)] = \mathbb{E}[\sum_j X^2_{t_j}(t_{j+1} - t_j)] = \mathbb{E}[\int_0^t X^2_s ds].
\]

Let us show (3). We already know that the process \( I_t(X) \) is continuous. From Itô isometry it follows that \( \mathbb{E}[I_t^2(X)] < \infty \). It remains to show that it is a martingale. Thus fix \( s < t \). Define \( t_n = t \) and define \( j_0 = \max\{j : t_j \leq s\} \).

\[
\mathbb{E}[I_t(X)|F_s] = \mathbb{E}[\sum_{j \leq n-1} X_{t_j}(B_{t_{j+1}} - B_{t_j})|F_s] \\
= \mathbb{E}[\sum_{j \leq j_0-1} X_{t_j}(B_{t_{j+1}} - B_{t_j})|F_s] + \mathbb{E}[X_{t_{j_0}}(B_s - B_{t_{j_0}})|F_s] \\
+ \mathbb{E}[X_{t_{j_0}}(B_{t_{j_0+1}} - B_s)|F_s] + \mathbb{E}[\sum_{j > j_0} X_{t_j}(B_{t_{j+1}} - B_{t_j})|F_s] \\
= \mathbb{E}[\sum_{j \leq j_0-1} X_{t_j}(B_{t_{j+1}} - B_{t_j})|F_s] + \mathbb{E}[X_{t_{j_0}}(B_s - B_{t_{j_0}})|F_s] \\
= I_s(X).
\]

(think about justifying last two equalities).

\( \square \)

2 Constructing Itô integral for general square integrable processes

The idea for defining Itô integral \( \int X dB \) for general processes in \( \mathcal{L}_2 \) is to approximate \( X \) by simple processes \( X^{(n)} \) and define \( \int X dB \) as a limit of \( \int X^{(n)} dB \), which we have already defined.

For this purpose we need to show that we can indeed approximate \( X \) with simple processes appropriately. We do this in 3 steps.

Step 1.

Proposition 1. Suppose \( X \in \mathcal{L}_2 \) is an a.s. bounded continuous process in the sense \( \exists M \) s.t. \( \mathbb{P}(\omega : \sup_{t \geq 0} |X_t(\omega)| \leq M) = 1 \). Then for every \( T > 0 \) there
exists a sequence of simple processes $X^n \in \mathcal{L}_2^0$ such that

$$
\lim_n \mathbb{E}\left[\int_0^T (X^n_t - X_t)^2 dt\right] = 0.
$$

(5)

**Proof.** Fix a sequence of partitions $\Pi_n = \{t^n_j\}$ of $[0, T]$ such that $\Delta_n = \max(t^n_{j+1} - t^n_j) \to 0$ as $n \to \infty$. Given process $X$, consider the modified process $X^n_t = X^n_{t^n_j}$ for all $t \in [t^n_j, t^n_{j+1})$. This process is simple and is adapted to $\mathcal{F}_t$. Since $X$ is a.s. continuous, then a.s. $X_t(\omega) = \lim_{n \to \infty} X^n_t(\omega)$ (notice that we are using left-continuity part of continuity). We conclude that a sequence of measurable functions $X^n : \Omega \times [0, T] \to \mathbb{R}$ a.s. converges to $X : \Omega \times [0, T] \to \mathbb{R}$. On the other hand $\mathbb{P}(\omega : \sup_{t \leq T} |X^n_t(\omega)| \leq M) = 1$. Using Bounded Convergence Theorem, the a.s. convergence extends to integrals: $\mathbb{E}\left[\int_0^T (X^n_t - X_t)^2 dt\right] \to 0$. □

**Step 2.**

**Proposition 2.** Suppose $X \in \mathcal{L}_2$ is a bounded, but not necessarily continuous process: $|X| \leq M$ a.s. For every $T > 0$, there exists a sequence of a.s. bounded continuous processes $X_n$ such that

$$
\lim_n \mathbb{E}\left[\int_0^T (X^n_t - X_t)^2 dt\right] = 0.
$$

(6)

**Proof.** We use a certain "regularization" trick to turn a bounded process into a bounded continuous approximation. Let $X^n_t = n \int_{t-1/n}^t X_s ds$. We have $|X^n| \leq n(1/n)M = M$ and $|X^n_t - X^n_{t'}| \leq 2n|t' - t|M$ (verify this), implying that $X^n_t$ is a.s. bounded continuous. Since $X_t$ is a.s. Riemann integrable, then for almost all $\omega$, the set of discontinuity points of of $X_t(\omega)$ has measure zero and for all continuity points $t$ by Fundamental Theorem of Calculus, we have $\lim_{n \to \infty} X^n_t(\omega) = X_t(\omega)$. We conclude that $X^n : \Omega \times [0, T] \to \mathbb{R}$ converges a.s. to $X$ on the same domain. Applying the Bounded Convergence Theorem we obtain the result. □

**Step 3.**

**Proposition 3.** Suppose $X \in \mathcal{L}_2$. For every $T > 0$ there exists a sequence of a.s. bounded processes $X_n \in \mathcal{L}_2$ such that

$$
\lim_n \mathbb{E}\left[\int_0^T (X^n_t - X_t)^2 dt\right] = 0.
$$

(7)
Proof. Define $X^n$ by $X^n_t = X_t$ when $-n \leq X_t \leq n$, $X^n_t = -n$, when $X_t < -n$ and $X^n_t = n$, when $X_t > n$. We have $X^n \to X$ a.s. w.r.t both $\omega$ and $t \in [0, T]$. Also $|X^n_t| \leq |X_t|$ implying

$$\int_0^T (X^n_t - X_t)^2 dt \leq 2 \int_0^T (X^n_t)^2 dt + 2 \int_0^T X_t^2 dt \leq 4 \int_0^T X_t^2 dt.$$ 

Since $\mathbb{E}[\int_0^T X_t^2 dt] < \infty$, then applying Dominated Convergence Theorem, we obtain the result.

Exercise 1. Establish (7) by applying instead Monotone Convergence Theorem.

3 Additional reading materials

- Karatzas and Shreve [1].
- Øksendal [2], Chapter III.

References

