Quadratic variation property of Brownian motion

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1 Unbounded variation of a Brownian motion

Any sequence of values $0 < t_0 < t_1 < \cdots < t_n < T$ is called a partition $\Pi = \Pi(t_0, \ldots, t_n)$ of an interval $[0, T]$. Given a continuous function $f : [0, T] \to \mathbb{R}$ its total variation is defined to be

$$LV(f) \triangleq \sup_{\Pi} \sum_{1 \leq k \leq n} |f(t_k) - f(t_{k-1})|,$$

where the supremum is taken over all possible partitions $\Pi$ of the interval $[0, T]$ for all $n$. A function $f$ is defined to have bounded variation if its total variation is finite.

**Theorem 1.** Almost surely no path of a Brownian motion has bounded variation for every $T \geq 0$. Namely, for every $T$

$$\mathbb{P}(\omega : LV(B(\omega)) < \infty) = 0.$$

The main tool is to use the following result from real analysis, which we do not prove: if a function $f$ has bounded variation on $[0, T]$ then it is differentiable almost everywhere on $[0, T]$. We will now show that quite the opposite is true.

**Proposition 1.** Brownian motion is almost surely nowhere differentiable. Specifically,

$$\mathbb{P}(\forall t \geq 0 : \limsup_{h \to 0} \left| \frac{B(t+h) - B(t)}{h} \right| = \infty) = 1.$$
Proof. Fix $T > 0, M > 0$ and consider $A(M, T) \subset C[0, \infty)$ – the set of all paths $\omega \in C[0, \infty)$ such that there exists at least one point $t \in [0, T]$ such that

$$\limsup_{h \to 0} \left| \frac{B(t + h) - B(t)}{h} \right| \leq M.$$  

We claim that $\mathbb{P}(A(M, T)) = 0$. This implies $\mathbb{P}(\bigcup_{M \geq 1} A(M, T)) = 0$ which is what we need. Then we take a union of the sets $A(M, T)$ with increasing $T$ and conclude that $B$ is almost surely nowhere differentiable on $[0, \infty)$. If $\omega \in A(M, T)$, then there exists $t \in [0, T]$ and $n$ such that $|B(s) - B(t)| \leq 2M|s - t|$ for all $s \in (t - \frac{2}{n}, t + \frac{2}{n})$. Now define $A_n \subset C[0, \infty)$ to be the set of all paths $\omega$ such that for some $t \in [0, T]$

$$|B(s) - B(t)| \leq 2M|s - t|$$

for all $s \in (t - \frac{2}{n}, t + \frac{2}{n})$. Then

$$A_n \subset A_{n+1} \tag{1}$$

and

$$A(M, T) \subset \bigcup_n A_n. \tag{2}$$

Find $k = \max\{j : \frac{j}{n} \leq t\}$. Define

$$Y_k = \max\{|B(\frac{k+2}{n}) - B(\frac{k+1}{n})|, |B(\frac{k+1}{n}) - B(\frac{k}{n})|, |B(\frac{k}{n}) - B(\frac{k-1}{n})|\}.$$  

In other words, consider the maximum increment of the Brownian motion over these three short intervals. We claim that $Y_k \leq 6M/n$ for every path $\omega \in A_n$.

To prove the bound required bound on $Y_k$ we first consider

$$|B(\frac{k+2}{n}) - B(\frac{k+1}{n})| \leq |B(\frac{k+2}{n}) - B(t)| + |B(t) - B(\frac{k+1}{n})|$$

$$\leq 2M\frac{2}{n} + 2M\frac{1}{n}$$

$$\leq \frac{6M}{n}.$$  

The other two differences are analyzed similarly.

Now consider event $B_n$ which is the set of all paths $\omega$ such that $Y_k(\omega) \leq 6M/n$ for some $0 \leq k \leq Tn$. We showed that $A_n \subset B_n$. We claim that $\lim_n \mathbb{P}(B_n) = \frac{2}{\sqrt{n}}$. 

0. Combining this with (1), we conclude $\mathbb{P}(A_n) = 0$. Combining with (2), this will imply that $\mathbb{P}(A(M, T)) = 0$ and we will be done.

Now to obtain the required bound on $\mathbb{P}(B_n)$ we note that, since the increments of a Brownian motion are independent and identically distributed, then

$$
\mathbb{P}(B_n) \leq \sum_{0 \leq k \leq T_n} \mathbb{P}(Y_k \leq 6M/n)
$$

$$
\leq Tn\mathbb{P}(\max\{|B(\frac{3}{n}) - B(\frac{2}{n})|, |B(\frac{2}{n}) - B(\frac{1}{n})|, |B(\frac{1}{n}) - B(0)|\} \leq 6M/n)
$$

$$
= Tn[\mathbb{P}(|B(\frac{1}{n})| \leq 6M/n)]^3. \tag{3}
$$

Finally, we just analyze this probability. We have

$$
\mathbb{P}(|B(\frac{1}{n})| \leq 6M/n) = \mathbb{P}(|B(1)| \leq 6M/\sqrt{n}).
$$

Since $B(1)$ which has the standard normal distribution, its density at any point is at most $1/\sqrt{2\pi}$, then we have that this probability is at most $(2(6M)/\sqrt{2\pi n})$. We conclude that the expression in (3) is, ignoring constants, $O(n(1/\sqrt{n})^3) = O(1/\sqrt{n})$ and thus converges to zero as $n \to \infty$. We proved $\lim_n \mathbb{P}(B_n) = 0$.

\section{Bounded quadratic variation of a Brownian motion}

Even though Brownian motion is nowhere differentiable and has unbounded total variation, it turns out that it has bounded \emph{quadratic} variation. This observation is the cornerstone of Ito calculus, which we will study later in this course.

We again start with partitions $\Pi = \Pi(t_0, \ldots, t_n)$ of a fixed interval $[0, T]$, but now consider instead

$$
Q(\Pi, B) \triangleq \sum_{1 \leq k \leq n} (B(t_k) - B(t_{k-1}))^2.
$$

where, we make (without loss of generality) $t_0 = 0$ and $t_n = T$. For every partition $\Pi$ define

$$
\Delta(\Pi) = \max_{1 \leq k \leq n} |t_k - t_{k-1}|.
$$

\textbf{Theorem 2.} Consider an arbitrary sequence of partitions $\Pi_i, i = 1, 2, \ldots$. Suppose $\lim_{i \to \infty} \Delta(\Pi_i) = 0$. Then

$$
\lim_{i \to \infty} \mathbb{E}[(Q(\Pi_i, B) - T)^2] = 0. \tag{4}
$$
Suppose in addition \( \lim_{i \to \infty} i^2 \Delta(\Pi_i) = 0 \) (that is the resolution \( \Delta(\Pi_i) \) converges to zero faster than \( 1/i^2 \)). Then almost surely
\[
Q(\Pi_i, B) \to T. \tag{5}
\]

In words, the standard Brownian motion has almost surely finite quadratic variation which is equal to \( T \).

**Proof.** We will use the following fact. Let \( Z \) be a standard Normal random variable. Then \( \mathbb{E}[Z^4] = 3 \) (cute, isn’t it?). The proof can be obtained using Laplace transforms of Normal random variables or integration by parts, and we skip the details.

Let \( \theta_i = (B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1}) \). Then, using the independent Gaussian increments property of Brownian motion, \( \theta_i \) is a sequence of independent zero mean random variables. We have
\[
Q(\Pi_i) - T = \sum_{1 \leq i \leq n} \theta_i.
\]

Now consider the second moment of this difference
\[
\mathbb{E}(Q(\Pi_i) - T)^2 = \sum_{1 \leq i \leq n} \mathbb{E}(B(t_i) - B(t_{i-1}))^4 - 2 \sum_{1 \leq i \leq n} \mathbb{E}(B(t_i) - B(t_{i-1}))^2(t_i - t_{i-1}) + \sum_{1 \leq i \leq n} (t_i - t_{i-1})^2.
\]

Using the \( \mathbb{E}[Z^4] = 3 \) property, this expression becomes
\[
\sum_{1 \leq i \leq n} 3(t_i - t_{i-1})^2 - 2 \sum_{1 \leq i \leq n} (t_i - t_{i-1})^2 + \sum_{1 \leq i \leq n} (t_i - t_{i-1})^2
\]
\[
= 2 \sum_{1 \leq i \leq n} (t_i - t_{i-1})^2
\]
\[
\leq 2 \Delta(\Pi_i) \sum_{1 \leq i \leq n} (t_i - t_{i-1})
\]
\[
= 2 \Delta(\Pi_i) T.
\]

Now if \( \lim_{i} \Delta(\Pi_i) = 0 \), then the bound converges to zero as well. This establishes the first part of the theorem.

To prove the second part identify a sequence \( \epsilon_i \to 0 \) such that \( \Delta(\Pi_i) = \epsilon_i/i^2 \). By assumption, such a sequence exists. By Markov’s inequality, this is
bounded by
\[ \mathbb{P}((Q(\Pi_i) - T)^2 > 2\epsilon_i) \leq \frac{\mathbb{E}(Q(\Pi_i) - T)^2}{2\epsilon_i} \leq \frac{2\Delta(\Pi_i)T}{2\epsilon_i} = \frac{T}{i^2} \] (6)

Since \( \sum_i \frac{T}{i^2} < \infty \), then the sum of probabilities in (6) is finite. Then applying the Borel-Cantelli Lemma, the probability that \((Q(\Pi_i) - T)^2 > 2\epsilon_i\) for infinitely many \(i\) is zero. Since \(\epsilon_i \to 0\), this exactly means that almost surely, \(\lim_i Q(\Pi_i) = T\).

3 Additional reading materials


References
