Traffic

Forecast: Number of cars will increase further
Fact: Infrastructure will not be enhanced to the same extent
Remedy: Improve the efficiency of traffic by other means

Effective Route Guidance in Traffic Networks

Lectures developed by
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2002 Urban Mobility Study
(http://mobility.tamu.edu/ums)

“The bad news is that even if transportation officials do all the right things, the likely effect is that congestion will continue to grow . . .”

• Total congestion “bill” in 2000 was $67.5 billion
  (= 3.6 billion hours delay + 5.7 billion gallons gas)

<table>
<thead>
<tr>
<th></th>
<th>1982</th>
<th>2000</th>
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<tr>
<td>time penalty for peak period travelers</td>
<td>16 hours</td>
<td>62 hours</td>
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Problem

People travel (between 6% and 19%) too much because of an unfavorable selection of their route.

(Beccaria & Bolelli 1992, Lösch 1995)

Outline

• Lecture 1
  Route Guidance; User Equilibrium; System Optimum; User Equilibria in Networks with Capacities.
• Lecture 2
  Constrained System Optimum; Dantzig-Wolfe Decomposition; Constrained Shortest Paths; Computational Results.

The Context

• Olaf Jahn (Research Assistant).
• Rolf H. Möhring (Principal Investigator).
  Collaboration with and support by DaimlerChrysler, Berlin.
• Nicolas Stier (Research Assistant).
• Andreas S. Schulz (Principal Investigator).
  Supported by General Motors Innovation Grant and SMA.
Shortest Path Routing

Improved network performance, but . . .

(Kaufman et al. 1991, Lee 1994)

Potential Remedies

• Toll systems
• Dynamic traffic signal control
• Park and Ride
• Traveller information systems

Shortest Path Routing II

. . . the same simulations show the performance decreases as soon as many cars use the system.

Proposed Solutions

• Multiple path routing:
  – k shortest paths
  – random perturbation

• Feedback control:
  – iterative computation of shortest paths

• Traffic assignment:
  – user equilibrium
  – system optimum
  – a new approach

In-Car Navigation Systems
Modeling Assumptions

**Reality**

- microscopic
  → individual vehicles
  → exact position, speed

- dynamic
  → consider time
  → on a single point at any time

- on-line
  → additional input over time

**Our Model**

- macroscopic
  → one abstract measure
  → traffic flow

- static
  → time independent
  → simultaneously at any point of the path

- off-line
  → all data known in advance

<table>
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<tr>
<th>selfish users</th>
<th>central planner</th>
<th>the goal</th>
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<tbody>
<tr>
<td>optimize own travel time</td>
<td>optimize system welfare</td>
<td>fair, efficient</td>
</tr>
<tr>
<td>fair, not efficient</td>
<td>efficient, not fair</td>
<td>fair, efficient</td>
</tr>
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</table>

Representation of the Road Network

How much can one gain?

- Study worst-case ratios between guided / unguided traffic

- Without guidance: use game theory to predict traffic
  (Wardrop 1952)

- Users’ behavior modeled as user equilibrium (Nash eq.)

- Price of anarchy is a measure of user equilibrium performance
  (Papadimitriou 2001)

Flows

- Different drivers have different origins and destinations

- Flows on paths: $f_P$ is the traffic along path $P$

- Flow on arcs:
  $$f_a = \sum_{P \in a} f_P$$
**The Traffic Model**

- Directed graph $G = (V, A)$, $k$ demands $(o_i, d_i)$ with rate $r_i$
- Flows on paths $f_P$. Can be non-integral.
- Traversal times: latency functions $t_a(\cdot)$
  - continuous and nondecreasing
  - belong to a given set $\mathcal{L}$ (e.g. linear)
- The total travel time of a flow is:
  \[
  C(f) := \sum_{a \in A} t_a(f_a)f_a
  \]

**Traversal Time Functions**

- Traversal time of an arc $a$ depends on the flow $f_a$ on $a$
- Dependence captured by the function $t_a(f_a)$
- Travel time along path $P$ is denoted by $t_P(f) = \sum_{a \in P} t_a(f_a)$

**System Optimum**

Convex Multicommodity Min-Cost Flow Problem

\[
\begin{align*}
& \min \sum_{a \in A} t_a(f_a)f_a \\
& \text{s.t.} \sum_{P \ni a} f_P = f_a \quad \text{for all } a \in A \\
& \quad \sum_{P \in \mathcal{P}_i} f_P = r_i \quad \text{for all } i = 1, \ldots, k \\
& \quad f_P \geq 0 \quad \text{for all } P \in \mathcal{P}
\end{align*}
\]

where

- $\mathcal{P}_i$ : set of paths from $o_i$ to $d_i$
- $\mathcal{P} = \bigcup_i \mathcal{P}_i$

**Example of SO**

(Pigou 1920)

\[
\text{SO} = \min f_a + f_b^2
\]

\[
\text{s.t. } f_a + f_b = 1 \\
\quad f_a, f_b \geq 0
\]

**The Traffic Model**

- Directed graph $G = (V, A)$, $k$ demands $(o_i, d_i)$ with rate $r_i$
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  \]

2 \cdot 4 + 2 \cdot 1 = 10

2 \cdot 3 = 6
Braess Paradox

- **UE non-monotone with network improvements** (Braess 1968)

![Network Diagram](image)

Example of SO

(Pigou 1920)

\[
SO = \min f_a + f_b^2 = \min f_b^2 + 1 - f_b = 3/4 \quad \text{and} \quad f_a = \frac{1}{2}, \quad f_b = \frac{1}{2}
\]

\[
\text{s.t. } f_a + f_b = 1 \quad \text{s.t.} \quad 0 \leq f_b \leq 1 \quad f_a, f_b \geq 0
\]

User Equilibrium

**Definition**: A flow is a **UE** iff nobody can switch to a path with smaller travel time.

- Travel times of users between the same OD-pair are equal
- **UE** always exists and is unique  
  (Beckmann et al. 1956)

![User Equilibrium Diagram](image)
**Networks with Capacities**

- Latencies model capacity only implicitly

What is the impact of having explicit capacities on arcs?

- Introduce capacities as hard constraints

- Straightforward to define SO

- What is now a UE?

**Braess Paradox**

- UE non-monotone with network improvements

(Braess 1968)

**Equilibria in Networks with Capacities**

**Definition**: A flow is a capacitated UE iff nobody can switch to a path with smaller travel time and residual capacity

- Travel times for users of same demand may differ (were constant w/o cap.)

- There may be multiple equilibria (UE w/o cap. was unique)

- How good is the best / worst eq.?

**Price of Anarchy measures impact of lack of Central Coordination**

(Papadimitriou 2001)

\[
\gamma := \max_{\text{inst.}} \frac{C(\text{UE})}{C(\text{SO})}
\]

- In general, \( \gamma \) unbounded

(Roughgarden & Tardos 2000)

- If latencies are in \( \mathcal{L} \), \( \gamma \leq \alpha(\mathcal{L}) \), where \( \alpha(\mathcal{L}) \) depends only on \( \mathcal{L} \)

In particular, \( \alpha(\{\text{linear latencies}\}) = 4/3 \)

(Roughgarden & Tardos 2000)

(Roughgarden 2002)

- Pigou’s and Braess’ examples are worst possible

**Multiple Equilibria**

with costs of: \( 4 \quad 4M \quad \frac{4M}{1+M} \)

Worst UE can be unbounded!
Network with Capacities: Guarantees

**Theorem.** For any instance of a network with capacities with latencies in $L$, we have

$$C(\text{BUE}) \leq \alpha(L) C(\text{SO})$$

In particular, if latencies are linear, $C(\text{BUE}) \leq \frac{4}{3} C(\text{SO})$

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**Proof (for the Linear Case)**

- Assume $t_a(f_a) = q_a f_a + r_a$ for all $a$ and let $f = \text{BUE}$

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Convex Optimization Review

- Let $z$ be a continuously differentiable and convex function on a convex set.
- Then $x^*$ is a global minimum of $z$ iff
  - the gradient along all feasible directions is non-negative

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Beckmann UE

- **Space of UE** non-convex: Difficult to get **Best UE**
- Instead, **Beckmann UE (BUE)** is
  $$\min \sum_{a \in A} \int_0^{f_a} t_a(x)dx$$
  subject to $f$ feasible flow
  capacity constraints

- Opt. Cond. **BUE**: $g$ feasible direction $\Rightarrow \sum_a g_a t_a(f_a) \geq 0$

---

Beckmann UE is an Equilibrium

**Lemma.** $f$ is a BUE $\Rightarrow f$ is a UE

**Proof:**

- Suppose $f$ is not a UE $\Rightarrow \exists$ two paths $P, Q$ s.t. flow can be re-routed from $P$ to $Q$ and $t_P(f) > t_Q(f)$
- $P$ and $Q$ define a circulation $g$ which is a feasible direction
- $\sum_a g_a t_a(f_a) < 0 \Rightarrow f$ is not a BUE

But **BUE** is not necessarily the best equilibrium

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Networks with Capacities
Non-convexity of UE

![Diagram showing non-convexity of UE](image)

**Proof (for the Linear Case)**

- Assume \( t_a(f_a) = q_a f_a + r_a \) for all \( a \) and let \( f = \text{BUE} \)
- Let \( C^L(x) = \sum_a x_a t_a(f_a) \)
- For all flows \( x \): \( C(f) \leq C^L(x) \)
  (same as \( \sum_a (x_a - f_a) t_a(f_a) \geq 0 \), the condition for \( \text{BUE} \))

Best vs. Beckmann

- The \( \text{BUE} \) does not need to be **best UE**:

![Diagram comparing BUE vs. best UE](image)

\( C(\text{BUE}) = 7 \) and \( C(\text{best UE}) = C(\text{SO}) = 6.875 \)

Proof (for the Linear Case)

- Assume \( t_a(f_a) = q_a f_a + r_a \) for all \( a \) and let \( f = \text{BUE} \)
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- \( C^L(x) = \sum_a x_a (q_a f_a + r_a) \leq \sum_a x_a (q_a x_a + r_a) + \frac{1}{4} \sum_a f_a^2 q_a \)
  because \( (x - f/2)^2 \geq 0 \)

Review

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<tr>
<th>No capacities</th>
<th>With capacities</th>
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<tr>
<td><strong>UE unique</strong></td>
<td>Set of <strong>UE</strong> may be non-convex</td>
</tr>
<tr>
<td><strong>UE/SO \geq \alpha(\mathcal{L})</strong></td>
<td><strong>UE/SO</strong> unbounded</td>
</tr>
<tr>
<td><strong>UE/SO \leq \alpha(\mathcal{L})</strong></td>
<td><strong>BUE/SO \leq \alpha(\mathcal{L})</strong></td>
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- Assume \( t_a(f_a) = q_a f_a + r_a \) for all \( a \) and let \( f = \text{BUE} \)
- Let \( C^L(x) = \sum_a x_a t_a(f_a) \)
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- \( C^L(x) = \sum_a x_a (q_a f_a + r_a) \leq \sum_a x_a (q_a x_a + r_a) + \frac{1}{4} \sum_a f_a^2 q_a \)
  because \( (x - f/2)^2 \geq 0 \)
- Last, \( \sum_a x_a (q_a x_a + r_a) + \frac{1}{4} \sum_a f_a^2 q_a \leq C(x) + \frac{1}{4} C(f) \)
  \[ \Rightarrow \quad \frac{1}{4} C(f) \leq C(x) \]