16.30/31 Homework Assignment #4

Goals: Modal analysis, transfer matrices, controllability and observability (part 1), linear system theory

1. Consider the system with two states, and the state-space model matrices given by:

\[ A = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ K \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \]

where \( K \in \mathbb{R} \) is a parameter to be specified.

(a) Find the transfer function \( G(s) \) for the system. Discuss the structure of \( G(s) \) for various values of \( K \).

(b) Form the observability matrix for the system. Is the system observable for all values of \( K \)?

(c) Form the controllability matrix for the system. Is the system controllable for all values of \( K \)?

(d) Compare your observations in parts (b) and (c) with those in part (a).

2. Given the transfer function from input \( u(t) \) to output \( y(t) \),

\[ \frac{Y(s)}{U(s)} = \frac{s^2 - 4s + 3}{(s^2 + 6s + 8)(s^2 + 25)} \]

(a) Develop a state space model for this transfer function, in the standard form

\[ \dot{x} = Ax + Bu \\
\quad y = Cx + Du. \]

(b) Suppose that zero input is applied, such that \( u = 0 \). Perform a modal analysis of the state response for this open-loop system. Your analysis should include the nature of the time response for each mode, as well as how each element of the state vector \( x = [x_1 \cdots x_n]^T \) contributes to that mode. Which mode dominates the time response? You may use Matlab to assist in your analysis.

(c) Now suppose that input of the form \( u = Ky \) is applied, where \( K = -15 \). Repeat the modal analysis of part (b) for this closed-loop system. (We will talk much more about this type of feedback later in the course.)
3. Given the MIMO system,

\[
G(s) = \begin{bmatrix}
\frac{7}{s^2 + 2} & \frac{2s + 8}{s^2 + 5s + 6} \\
\frac{3s + 15}{s^2 + 7s + 10} & \frac{5}{s + 3}
\end{bmatrix}
\]

(a) Develop a state space model using the technique described at the bottom of page 8–5. Using Matlab, verify that this is not a minimal realization.

(b) Develop a state space model using Gilbert’s realization method on page 8–8. Using Matlab, verify that this is a minimal realization.

*Hint:* It is easy to confirm that each state space model will give the same transfer function matrix.

4. (16.31 required/16.30 extra credit) Consider the homogeneous system

\[
\dot{x}(t) = A(t)x(t)
\]

with initial condition \(x(t_0) = x_0\). The general solution to this differential equation is given by

\[
x(t) = \Phi(t, t_0)x(t_0)
\]

where \(\Phi(t_1, t_1) = I\). Prove that the following properties of the state transition matrix are true:

(a) \(\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)\), regardless of the order of the \(t_i\)

(b) \(\Phi(t, \tau) = \Phi(\tau, t)^{-1}\)

(c) \(\frac{\partial}{\partial t} \Phi(t, t_0) = A(t)\Phi(t, t_0)\)