16.30/31 Feedback Control Systems
Overview of Nonlinear Control Synthesis

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An overview of nonlinear control design methods

- Extend applicability of linear design methods:
  - Gain scheduling
  - Integrator anti-windup schemes

- Geometric control
  - Feedback linearization
  - Dynamics inversion
  - Differential flatness

- Adaptive control
  - Neural network augmentation

- Lyapunov-based methods/Contraction theory
  - Control Lyapunov Functions
  - Sliding mode control
  - Backstepping

- Computational/logic approaches
  - Hybrid systems
  - Model Predictive Control
Gain scheduling

- Nonlinear system: \( \dot{x} = f(x, u) \).

- Choose \( n \) equilibrium points, i.e., \((x_i^*, u_i^*)\), such that \( f(x_i^*, u_i^*) = 0 \), \( i = 1, \ldots, n \).

- For each of these equilibria, linearize the system and design a “local” control law \( u_i(x) = u_i^* - K(x - x_i^*) \) for the linearization.

- A global control law consists of:
  - Choose the right control law, as a function of the state: \( i = \sigma(x) \)
  - Use that control law: \( u(x) = u_{\sigma(x)}(x) \)
Control Lyapunov functions

- Nonlinear system: \( \dot{x} = f(x) + g(x)u \), with equilibrium at \( x = 0 \)

- A function \( V : x \mapsto V(x) \) is a Control Lyapunov Function if
  - It is positive definite
  - \( V(0) = 0 \).
  - It is always possible to find \( u \) such that
    \[
    \dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u \leq 0.
    \]
  - If \( V \) is a CLF, it is always possible to design a control law ensuring \( \dot{V} \leq 0 \)!
Differential Flatness

A dynamical system

\[
\frac{d x}{d t} = f(x, u),
\]

\[z = h(x, u).\]

is said to be differentially flat, with flat output \(z\), if one can compute the state and input trajectories as a function of the flat outputs and a finite number of its derivatives, i.e., if one can find a map \(\Xi\) such that

\[(x, u) = \Xi(z, \dot{z}, \ldots, z^{(l)}).\]

Differential flatness can be shown to be equivalent to (dynamic) feedback linearizability. (But more intuitive.)
A differentially-flat model of aircraft dynamics 1/4

- Given a reference trajectory, $p_d$, we can compute the reference velocity $\dot{p}_d$, and the reference acceleration $\ddot{p}_d$.

- Based on these, and assuming coordinated flight (and hence $\dot{p}_d \neq 0$) we can get a set of reference wind axes:
  - The $x_w$ axis is aligned with the velocity vector $p_d$, i.e., $x_w := \dot{p}_d / \|\dot{p}_d\|$.
  - The acceleration can be written as
    \[ \ddot{p}_d = g + f_i / m, \]
    where $f_i$ is the aerodynamic/propulsive force in inertial frame, and $g$ is the gravity vector.
  - The main sources of forces for an airplane are the engine thrust and lift. Both are approximately contained within the symmetry plane of the aircraft. Hence, the $z_w$ axis is chosen such that the $(x_w, z_w)$ plane contains $f_i$.
  - The $y_w$ axis is chosen to complete a right-handed orthonormal triad.
Assume that we can control independently:
  
  - The tangential acceleration\(^1\) \(a_t\) along the wind velocity vector (\(x_w\) axis).
  - The normal acceleration \(a_n\) (along the \(z_w\) axis).
  - The roll rate \(\omega_1\) around the wind velocity vector (\(x_w\) axis).

Recall \(\ddot{p}_d = g + a_I = g + Ra_w\), where \(a_I\) is the acceleration in the inertial frame, \(a_w\) is the acceleration in the wind frame defined in the previous slide, and \(R\) is a rotation matrix, computed as a function of \(\dot{p}_d, \ddot{p}_d\).

Differentiating,

\[
\dddot{p}_d^{(3)} = \dot{R}a_w + R\dot{a}_w = R(\omega \times a_w) + R\dot{a}_w.
\]

In addition, since \(\dot{p}_d = VRe_1\) (coordinated flight), we also have

\[
\dddot{p}_d = g + Ra_w = \dot{V}Re_1 + VR(\omega \times e_1)
\]

\(^1\)Here and below acceleration is understood as “acceleration due to aerodynamic/propulsive forces.”
In coordinates, the second equation reads:

\[
\ddot{p}_d = R \begin{bmatrix} \dot{V} \\ V \omega_3 \\ -V \omega_2 \end{bmatrix},
\]

i.e., \( \omega_2 \) and \( \omega_3 \) can be computed from \( \dot{p} \) and \( \ddot{p} \).

Also,

\[
\frac{d}{dt^3} \begin{bmatrix} p_{d,1} \\ p_{d,2} \\ p_{d,3} \end{bmatrix} = R \begin{bmatrix} \omega_2 a_n + \dot{a}_t \\ \omega_3 a_t - \omega_1 a_n \\ -\omega_2 a_t + \dot{a}_n \end{bmatrix}
\]
Finally, we have

\[
\begin{bmatrix}
\dot{a}_t \\
\omega_1 \\
\dot{a}_n
\end{bmatrix}
= \begin{bmatrix}
-\omega_2 a_n \\
\omega_3 a_t / a_n \\
\omega_2 a_t
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/a_n & 0 \\
0 & 0 & 1
\end{bmatrix}
R^T \frac{d^3}{dt^3}
\begin{bmatrix}
p_{d,1} \\
p_{d,2} \\
p_{d,3}
\end{bmatrix}
\]

I.e., the system is differentially flat, with flat output \(p_d\), as long as

- \(V = \|\dot{p}_d\| \neq 0\): if the velocity is zero, then coordinated flight is not well defined.

- \(a_n = (I - \frac{1}{V^2} \dot{p}_d \dot{p}_d^T) (\ddot{p}_d - g) \neq 0\): if the normal acceleration is zero, the roll attitude is not well defined.
Incorporating aerodynamics and propulsive models

- We have derived a differentially flat system for aircraft dynamics, with flat output $p_d$ (position trajectory) and inputs: $(\dot{a}_t, \omega_1, \dot{a}_n)$.

- How can we control $a_t$, $a_n$ (or their derivatives)?
  - The wing lift is
    \[ L = \frac{1}{2m}\rho V^2 S C_{L\alpha} \alpha + a_{L0}, \]
    the drag is similarly computed.
  - The engine thrust $T$ is a function of the throttle setting $\delta_T$ and other variables.
  - The acceleration components in wind axes are given by
    \[ a_t = T(\delta_T) \cos \alpha - D(\alpha), \]
    \[ a_n = -T(\delta_T) \sin \alpha - L(\alpha). \]

- Compute, e.g., $\alpha$ and $\delta_T$, from the desired $a_t$, $a_n$.

- Rely on a CAS that tracks the commanded $\alpha$, $\delta_T$, and $\omega_1$. 

Adding feedback

- So far, we have shown that, given a reference trajectory, we can compute uniquely (modulo a $180^\circ$ roll rotation) the corresponding state and control input trajectories.

- What if the initial condition is not on the reference trajectory? What if there are disturbances that make the aircraft deviate from the trajectory? We need feedback.

- Let $p : t \mapsto p(t)$ be the actual position of the vehicle, and consider a system in which $p^{(3)} = u$, e.g.,

\[
\frac{d}{dt} \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & l & 0 \\ 0 & 0 & l \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} u
\]
Define the error $e = p - p_d$. The error dynamics are given by

$$
\frac{d}{dt} \begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left( u - p_d^{(3)} \right).
$$

If we set

$$
u = p_d^{(3)} - K[e, \dot{e}, \ddot{e}]^T,$$

where $k$ is a stabilizing control gain for the error dynamics, and compute $a_t, a_n, \omega_1$ from $(p, \dot{p}, \ddot{p}, u)$ (vs. $p_d$ and its derivatives)

then,

$$
\lim_{t \to +\infty} e(t) = \lim_{t \to +\infty} (p(t) - p_d(t)) = 0,
$$

as desired.
Some remarks

- Convergence assured “almost” globally, the control law breaks down if at any point $\dot{x} = 0$, or $a_n = 0$.

- A modification of this control law can ensure path tracking (vs. trajectory tracking), requiring less thrust control effort. See Hauser & Hindman '97.

- Some limitations:
  - Simplified aerodynamic/propulsive models.
  - Saturations are not taken into account (unbounded $a_n, a_t, \omega_1$).
  - Coordinated flight is an additional constraints, no control over roll: this model cannot account for, e.g., a split-S maneuver.