Choice Theory – A Synopsis

1. Basic Concepts:
   - 1. Choice
   - 2. Preference
   - 3. Utility

2. Weak Axiom of Revealed Preferences
3. Preference as a representation of choice
4. Ordinal Utility Representation
5. Continuity
Basic Concepts

- $X = \text{Set of Alternatives}$
  - Mutually exclusive
  - Exhaustive
- $A = \text{non-empty set of available alternatives}$
- Choice Function: $c : A \mapsto c(A) \subseteq A$.
  - $c(A)$ is non-empty
- Preference: A relation $\succeq$ on $X$ that is
  - complete: $\forall x, y \in X$, either $x \succeq y$ or $y \succeq x$;
  - transitive: $\forall x, y, z \in X$, $[x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z$.
- Utility Function: $U : X \rightarrow \mathbb{R}$

Choice Function

- $c : A \mapsto c(A) \subseteq A$
- It describes what alternatives DM may choose under each set of constraints
- Feasibility: $c(A) \subseteq A$.
- Exhaustive: $c(A)$ is non-empty
- Mutually exclusive: only one alternative is chosen
Preference

- Preference Relation: A relation \( \succeq \) on \( X \) s.t.
  - complete: \( \forall x, y \in X \), either \( x \succeq y \) or \( y \succeq x \);
  - transitive: \( \forall x, y, z \in X \), \( [x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z \).
- \( x \succeq y \) means: DM finds \( x \) at least as good as \( y \)
- Preferences do not depend on \( A \)!
- Strict Preference: \( x > y \leftrightarrow [x \succeq y \text{ and not } y \succeq x] \)
- Indifference: \( x \sim y \leftrightarrow [x \succeq y \text{ and } y \succeq x] \).
- Choice induced by preference:
  \[ c_>(A) = \{x \in A | x \succeq y \ \forall y \in A\} \]

Choice v. Preference

**Definition:** A choice function \( c \) is represented by \( \succeq \) iff \( c = c_\succ \).

**Theorem:** Assume that \( X \) is finite. A choice function \( c \) is represented by some preference relation \( \succeq \) if and only if \( c \) satisfies WARP.
Weak Axiom of Revealed Preference

**Axiom (WARP):** For all \(A, B \subseteq X\) and \(x, y \in A \cap B\), if \(x \in c(A)\) and \(y \in c(B)\), then \(x \in c(B)\).

- **WARP:** DM has well-defined preferences
  - That govern the choice
  - Don’t depend on the set \(A\) of feasible alternatives

Ordinal Utility Representation

**Ordinal Representation:** \(U : X \rightarrow \mathbb{R}\) is an ordinal representation of \(\succeq\) iff:
\[
x \succeq y \iff U(x) \geq U(y) \quad \forall x, y \in X.
\]

**Fact:** If \(U\) represents \(\succeq\) and \(f : \mathbb{R} \rightarrow \mathbb{R}\) is strictly increasing, then \(f \circ U\) represents \(\succeq\).

**Theorem:** Assume \(X\) is finite (or countable). A relation has an ordinal representation if and only if it is complete and transitive.

**Example:** Lexicographic preference relation on unit square does not have an ordinal representation.
Continuous Representation

**Definition:** A preference relation $\succeq$ is said to be continuous iff $\{y \mid y \succeq x\}$ and $\{y \mid x \succeq y\}$ are closed for every $x$ in $X$.

**Theorem:** Assume $X$ is a compact, convex subset of a separable metric space. A preference relation has a continuous ordinal representation if and only if it is continuous.

Indifference Sets of a Continuous Preference

- $l(x) = \{ y \mid x \sim y \}$
- $l(x)$ is closed.
- If
  - $x' \succ x \succ x''$
  - $\phi:[0,1] \to X$ continuous
  - $\phi(1)=x'$; $\phi(0)=x''$,
- Then, $\exists \ t \in [0,1]$ such that $\phi(t) \sim x$. 