Recursive Methods
• study Functional Equation (Bellman equation) with bounded and continuous $F$

• tools: contraction mapping and theorem of the maximum
Bellman Equation as a Fixed Point

• define operator

\[ T(f)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta f(y) \} \]

• \( V \) solution of BE \( \iff \) \( V \) fixed point of \( T \) [i.e. \( TV = V \)]

**Bounded Returns:**

• if \( \|F\| < B \) and \( F \) and \( \Gamma \) are continuous: \( T \) maps continuous bounded functions into continuous bounded functions

• bounded returns \( \Rightarrow \) \( T \) is a Contraction Mapping \( \Rightarrow \) unique fixed point

• many other bonuses
Our Favorite Metric Space

\[ S = \left\{ f : X \to R, \text{ } f \text{ is continuous, and } \|f\| \equiv \sup_{x \in X} |f(x)| < \infty \right\} \]

with

\[ \rho(f, g) = \|f - g\| \equiv \sup_{x \in X} |f(x) - g(x)| \]

**Definition.** A linear space \( S \) is complete if any Cauchy sequence converges. For a definition of a Cauchy sequence and examples of complete metric spaces see SLP.

**Theorem.** The set of bounded and continuous functions is Complete. See SLP.
Definition. Let $(S, \rho)$ be a metric space. Let $T : S \rightarrow S$ be an operator. $T$ is a contraction with modulus $\beta \in (0, 1)$

$$\rho(Tx, Ty) \leq \beta \rho(x, y)$$

for any $x, y$ in $S$. 

Introduction to Dynamic Optimization
**Contraction Mapping Theorem**

**Theorem (CMThm).** If $T$ is a contraction in $(S, \rho)$ with modulus $\beta$, then (i) there is a unique fixed point $s^* \in S$,

$$s^* = Ts^*$$

and (ii) iterations of $T$ converge to the fixed point

$$\rho(T^n s_0, s^*) \leq \beta^n \rho(s_0, s^*)$$

for any $s_0 \in S$, where $T^{n+1}(s) = T(T^n(s))$. 

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*Introduction to Dynamic Optimization*  
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for (i) **1st step:** construct fixed point \( s^* \)
take any \( s_0 \in S \) define \( \{s_n\} \) by \( s_{n+1} = Ts_n \) then

\[
\rho(s_2, s_1) = \rho(Ts_1, Ts_0) \leq \beta \rho(s_1, s_0)
\]

generalizing \( \rho(s_{n+1}, s_n) \leq \beta^n \rho(s_1, s_0) \) then, for \( m > n \)

\[
\rho(s_m, s_n) \leq \rho(s_m, s_{m-1}) + \rho(s_{m-1}, s_{m-2}) + \cdots + \rho(s_{n+1}, s_n)
\leq [\beta^{m-1} + \beta^{m-2} + \cdots + \beta^n] \rho(s_1, s_0)
\leq \beta^n [\beta^{m-n-1} + \beta^{m-n-2} + \cdots + 1] \rho(s_1, s_0)
\leq \frac{\beta^n}{1 - \beta} \rho(s_1, s_0)
\]

thus \( \{s_n\} \) is cauchy. hence \( s_n \to s^* \)
2nd step: show $s^* = Ts^*$

$$\rho(Ts^*, s^*) \leq \rho(Ts^*, s_n) + \rho(s^*, s_n) \leq \beta \rho(s^*, s_{n-1}) + \rho(s^*, s_n) \to 0$$

3nd step: $s^*$ is unique. $Ts_1^* = s_1^*$ and $s_2^* = Ts_2^*$

$$0 \leq a = \rho(s_1^*, s_2^*) = \rho(Ts_1^*, Ts_2^*) \leq \beta \rho(s_1^*, s_2^*) = \beta a$$

only possible if $a = 0 \Rightarrow s_1^* = s_2^*$.

Finally, as for (ii):

$$\rho(T^n s_0, s^*) = \rho(T^n s_0, Ts^*) \leq \beta \rho(T^{n-1} s_0, s^*) \leq \cdots \leq \beta^n \rho(s_0, s^*)$$
**Corollary.** Let $S$ be a complete metric space, let $S' \subset S$ and $S'$ close. Let $T$ be a contraction on $S$ and let $s^* = Ts^*$. Assume that

$$T(S') \subset S', \text{ i.e. if } s' \in S, \text{ then } T(s') \in S'$$

then $s^* \in S'$. Moreover, if $S'' \subset S'$ and

$$T(S') \subset S'', \text{ i.e. if } s' \in S', \text{ then } T(s') \in S''$$

then $s^* \in S''$. 


Blackwell’s sufficient conditions.
Let $S$ be the space of bounded functions on $X$, and $\|\cdot\|$ be given by the sup norm. Let $T : S \rightarrow S$. Assume that (i) $T$ is monotone, that is,

$$Tf(x) \leq Tg(x)$$

for any $x \in X$ and $g, f$ such that $f(x) \geq g(x)$ for all $x \in X$, and (ii) $T$ discounts, that is, there is a $\beta \in (0, 1)$ such that for any $a \in R_+$,

$$T(f + a)(x) \leq Tf(x) + a\beta$$

for all $x \in X$. Then $T$ is a contraction.
Proof. By definition
\[ f = g + f - g \]
and using the definition of \( \| \cdot \| \),
\[ f(x) \leq g(x) + \| f - g \| \]
then by monotonicity i)
\[ Tf \leq T(g + \| f - g \|) \]
and by discounting ii) setting \( a = \| f - g \| \)
\[ Tf \leq T(g) + \beta \| f - g \| . \]
Reversing the roles of \( f \) and \( g \):
\[ Tg \leq T(f) + \beta \| f - g \| \]
\[ \Rightarrow \| Tf - Tg \| \leq \beta \| f - g \| \]
Bellman equation application

\[(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}\]

Assume that \(F\) is bounded and continuous and that \(\Gamma\) is continuous and has compact range.

**Theorem.** \(T\) maps the set of continuous and bounded functions \(S\) into itself. Moreover \(T\) is a contraction.
Proof. That $T$ maps the set of continuous and bounded follow from the Theorem of Maximum (we do this next)
That $T$ is a contraction follows since $T$ satisfies the Blackwell sufficient conditions.
$T$ satisfies the Blackwell sufficient conditions. For monotonicity, notice that for $f \geq v$

$$Tv(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \}$$
$$= F(x, g(x)) + \beta v(g(x))$$
$$\leq \{ F(x, g(y)) + \beta f(g(x)) \}$$
$$\leq \max_{y \in \Gamma(x)} \{ F(x, y) + \beta f(y) \} = Tf(x)$$

A similar argument follows for discounting: for $a > 0$

$$T(v + a)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta (v(y) + a) \}$$
$$= \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \} + \beta a = T(v)(x) + \beta a.$$
Theorem of the Maximum

• want $T$ to map continuous function into continuous functions

\[(Tv)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \} \]

• want to learn about optimal policy of RHS of Bellman

\[G(x) = \arg \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \} \]

• First, continuity concepts for correspondences
• ... then, a few example maximizations
• ... finally, Theorem of the Maximum
assume $\Gamma$ is non-empty and compact valued (the set $\Gamma(x)$ is non empty and compact for all $x \in X$)

**Upper Hemi Continuity (u.h.c.) at $x$:** for any pair of sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$ and $x_n \in \Gamma(y_n)$ there exists a subsequence of $\{y_n\}$ that converges to a point $y \in \Gamma(x)$.

**Lower Hemi Continuity (l.h.c.) at $x$:** for any sequence $\{x_n\}$ with $x_n \to x$ and for every $y \in \Gamma(x)$ there exists a sequence $\{y_n\}$ with $x_n \in \Gamma(y_n)$ such that $y_n \to y$.

**Continuous at $x$:** if $\Gamma$ is both upper and lower hemi continuous at $x$
Max Examples

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]
\[ G(x) = \arg \max_{y \in \Gamma(x)} f(x, y) \]

**ex 1**: \( f(x, y) = xy; \ X = [-1, 1]; \ \Gamma(x) = X. \)

\[ G(x) = \begin{cases} \{-1\} & x < 0 \\ [-1, 1] & x = 0 \\ \{1\} & x > 0 \end{cases} \]
\[ h(x) = |x| \]
**ex 2:** \( f(x, y) = xy^2 \), \( X = [-1, 1] \); \( \Gamma(x) = X \)

\[
G(x) = \begin{cases} 
0 & x < 0 \\
[-1, 1] & x = 0 \\
\{ -1, 1 \} & x > 0 
\end{cases}
\]

\( h(x) = \max \{ 0, x \} \)
Theorem of the Maximum

Define:

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]

\[ G(x) = \text{arg max}_{y \in \Gamma(x)} f(x, y) \]

\[ = \{ y \in \Gamma(x) : h(x) = f(x, y) \} \]

**Theorem.** (Berge) Let \( X \subset \mathbb{R}^l \) and \( Y \subset \mathbb{R}^m \). Let \( f : X \times Y \rightarrow \mathbb{R} \) be continuous and \( \Gamma : X \rightarrow Y \) be compact-valued and continuous. Then \( h : X \rightarrow \mathbb{R} \) is continuous and \( G : X \rightarrow Y \) is non-empty, compact valued, and u.h.c.
lim max → max lim

**Theorem.** Suppose \( \{f_n(x, y)\} \) and \( f(x, y) \) are concave in \( y \) and \( f_n \to f \) in the sup-norm (uniformly). Define

\[
    g_n(x) = \arg \max_{y \in \Gamma(x)} f_n(x, y)
\]

\[
    g(x) = \arg \max_{y \in \Gamma(x)} f(x, y)
\]

then \( g_n(x) \to g(x) \) for all \( x \) (pointwise convergence); if \( X \) is compact then the convergence is uniform.
Monotonicity of \( v^* \)

**Theorem.** Assume that \( F(\cdot, y) \) is increasing, that \( \Gamma \) is increasing, i.e.

\[
\Gamma (x) \subset \Gamma (x')
\]

for \( x \leq x' \). Then, the unique fixed point \( v^* \) satisfying \( v^* = Tv^* \) is increasing. If \( F(\cdot, y) \) is strictly increasing, so is \( v^* \).
Proof

By the corollary of the CMThm, it suffices to show $Tf$ is increasing if $f$ is increasing. Let $x \leq x'$:

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

$$= F(x, y^*) + \beta f(y^*) \text{ for some } y^* \in \Gamma(x)$$

$$\leq F(x', y^*) + \beta f(y^*)$$

since $y^* \in \Gamma(x) \subset \Gamma(x')$

$$\leq \max_{y \in \Gamma(x')} \{F(x, y) + \beta f(y)\} = Tf(x')$$

If $F(\cdot, y)$ is strictly increasing

$$F(x, y^*) + \beta f(y^*) < F(x', y^*) + \beta f(y^*).$$
**Concavity (or strict) concavity of $v^*$**

**Theorem.** Assume that $X$ is convex, $\Gamma$ is concave, i.e. $y \in \Gamma(x)$, $y' \in \Gamma(x')$ implies that

$$y^\theta \equiv \theta y' + (1 - \theta) y \in \Gamma(\theta x' + (1 - \theta) x) \equiv \Gamma(x^\theta)$$

for any $x, x' \in X$ and $\theta \in (0, 1)$. Finally assume that $F$ is concave in $(x, y)$. Then, the fixed point $v^*$ satisfying $v^* = Tv^*$ is concave in $x$. Moreover, if $F(\cdot, y)$ is strictly concave, so is $v^*$. 

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Differentiability

- can’t use same strategy: space of differentiable functions is not closed
- many envelope theorems
- Formula: if \( h(x) \) is differentiable and \( y \) is interior then
  \[
  h'(x) = f_x(x, y)
  \]
  right value... but is \( h \) differentiable?
- one answer (Demand Theory) relies on f.o.c. and assuming twice differentiability of \( f \)
- won’t work for us since \( f = F(x, y) + \beta V(y) \) and we don’t even know if \( f \) is once differentiable! → going in circles
First a Lemma...

**Lemma.** Suppose \( v(x) \) is concave and that there exists \( w(x) \) such that \( w(x) \leq v(x) \) and \( v(x_0) = w(x_0) \) in some neighborhood \( D \) of \( x_0 \) and \( w \) is differentiable at \( x_0 \) (\( w'(x_0) \) exists) then \( v \) is differentiable at \( x_0 \) and \( v'(x_0) = w'(x_0) \).

**Proof.** Since \( v \) is concave it has at least one subgradient \( p \) at \( x_0 \):

\[
w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0)
\]

Thus a subgradient of \( v \) is also a subgradient of \( w \). But \( w \) has a unique subgradient equal to \( w'(x_0) \). \( \square \)
Now a Theorem

**Theorem.** Suppose $F$ is strictly concave and $\Gamma$ is convex. If $x_0 \in \text{int}(X)$ and $g(x_0) \in \text{int}(\Gamma(x_0))$ then the fixed point of $T$, $V$, is differentiable at $x$ and

$$V'(x) = F_x(x, g(x))$$

**Proof.** We know $V$ is concave. Since $x_0 \in \text{int}(X)$ and $g(x_0) \in \text{int}(\Gamma(x_0))$ then $g(x_0) \in \text{int}(\Gamma(x))$ for $x \in D$ a neighborhood of $x_0$ then

$$W(x) = F(x, g(x_0)) + \beta V(g(x_0))$$

and then $W(x) \leq V(x)$ and $W(x_0) = V(x_0)$ and $W'(x_0) = F_x(x_0, g(x_0))$ so the result follows from the lemma. □