Problem 1

a. Let $\tilde{X}$ denote the sum of the weight of the 100 sampled coins: $\tilde{X} = \sum_{i=1}^{100} X_i$. Now, $\tilde{X}$ must be distributed normally, because it is a linear combination of independent normal random variables. Then $E(\tilde{X}) = E\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} E(X_i) = 12800$, and $Var(\tilde{X}) = Var\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} Var(X_i) = 100$, since the coins are assumed to be independent. Thus, if the Master is honest, $\tilde{X} \sim N(12800, 100)$.

We must choose some remedy $R$, such that we punish the Master whenever $\tilde{X} \leq 12800 - R$, and we want the probability that an honest Master is punished be no greater than 0.01. So we want to find $R$ that satisfies

$$Pr\left(\tilde{X} \leq 12800 - R\right) = 0.01$$

$$Pr\left(\frac{\tilde{X} - 12800}{10} \leq -\frac{R}{10}\right) = 0.01$$

$$Pr\left(Z \leq -\frac{R}{10}\right) = 0.01$$

We can use our table to find that $-\frac{R}{10} = -2.33$, which implies $R = 23$.

b. Under the dishonest master, $\tilde{X} \sim N(12790, 100)$. For $R = 23$, the probability of catching this master is $Pr\left(\tilde{X} \leq 12777\right) = Pr\left(\frac{\tilde{X} - 12800}{10} \leq -\frac{12777 - 12800}{10}\right) = Pr(Z \leq -1.3) \approx 0.097$. Thus, the chance that this dishonest Master eludes punishment is greater than 90%.

Problem 2
a. \[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2, \] so \[ E(S^2) = E\left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2\right) \]

\[
E(S^2) = E\left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2\right) \\
= \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_i - \overline{X})^2\right) \\
= \frac{1}{n-1} E\left(\sum_{i=1}^{n} \left(X_i^2 - 2X_i \overline{X} + \overline{X}^2\right)\right) \\
= \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - 2\overline{X} \sum_{i=1}^{n} X_i + n\overline{X}^2\right) \\
= \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - 2n\overline{X}^2 + n\overline{X}^2\right) \\
= \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right) \\
= \frac{1}{n-1} \left(\sum_{i=1}^{n} E(X_i^2) - nE(\overline{X}^2)\right) \\
= \frac{1}{n-1} \left(n\sigma^2 + n\mu^2 - n\left(Var(\overline{X}) + \mu^2\right)\right) \\
= \frac{1}{n-1} \left(n\sigma^2 - n\frac{\sigma^2}{n}\right) \\
= \frac{n-1}{n-1} (\sigma^2) \\
= \sigma^2
\]

b. To show \( \overline{X} \xrightarrow{p} \mu \), we want to prove that for any \( \varepsilon > 0 \), as \( n \to \infty \),
\[
Pr\left(|\overline{X} - \mu| < \varepsilon\right) \to 1.
\]
We will denote the mean of a random sample of
size $n$ by $\bar{X}_n$.

\[
Pr\left(|\bar{X}_n - \mu| < \varepsilon\right) = Pr\left(-\varepsilon < \bar{X}_n - \mu < \varepsilon\right)
\]
\[
= 1 - 2 Pr\left(\bar{X}_n - \mu > \varepsilon\right)
\]
\[
= 1 - 2 Pr\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} > \frac{\varepsilon}{\sigma/\sqrt{n}}\right)
\]
\[
= 1 - 2 Pr\left(Z > \sqrt{n} \frac{\varepsilon}{\sigma}\right)
\]
\[
= 1 - 2 \left(1 - Pr\left(Z < \sqrt{n} \frac{\varepsilon}{\sigma}\right)\right)
\]
\[
= -1 + 2 Pr\left(Z < \sqrt{n} \frac{\varepsilon}{\sigma}\right)
\]

Now, as $n \to \infty$, for any $\varepsilon > 1$, $\Phi\left(\sqrt{n} \frac{\varepsilon}{\sigma}\right) \to 1$, so $Pr\left(|\bar{X}_n - \mu| < \varepsilon\right) \to -1 + 2 = 1$.

**Problem 3**

a. Expand the expression for the price of the stock:

\[
S_t = S_0 + \sum_{i=1}^{t} X_i
\]

Thus the change in the stock's price after 700 periods is

\[
\Delta S = S_{700} - S_0 = \sum_{i=1}^{700} X_i
\]

b. Define

\[
\bar{X} = \frac{1}{700} \sum_{i=1}^{700} X_i
\]

Since 700 is "large," the Central Limit Theorem implies that

\[
\frac{\sqrt{700} (\bar{X} - \mu)}{\sigma} \sim N(0, 1)
\]

\[
\mu = 0.39 (1) + 0.20 (0) + 0.41 (-1)
\]
\[
= 0.02
\]

\[
E (X_i^2) = 0.39 (1)^2 + 0.20 (0)^2 + 0.41 (-1)^2
\]
\[
= 0.80
\]

\[
\sigma^2 = 0.80 - (-0.02)^2
\]
\[
= 0.7996
\]

Therefore

\[
\bar{X} \sim N\left(-0.02, \frac{0.7996}{700}\right)
\]
c. The probability that the stock is up at least 10 over the first 700 periods can be calculated as follows:

\[
\Pr \left( \sum_{i=1}^{700} X_i \geq 10 \right) = \Pr \left( \bar{X} \geq \frac{1}{70} \right)
\]
\[
= \Pr \left( \frac{\bar{X} + 0.02}{\sqrt{0.7996/700}} \geq \frac{1}{70} + 0.02 \right)
\]
\[
= \Pr (Z \geq 1.01)
\]
\[
= 1 - \Pr (Z \leq 1.01)
\]
\[
\approx 0.156
\]

**Problem 4**

We can denote the weights of each of the hundred booklets as \(X_1, \ldots, X_{100}\). This is a random sample from a population with mean 1 and standard deviation 0.05. We want to know the probability that the sum of the \(X_i\) is greater than 100.4:

\[
\Pr \left( \sum_{i=1}^{100} X_i \geq 100.4 \right) = \Pr (\bar{X} \geq 1.004)
\]

The sample mean has the properties such that

\[
E(\bar{X}) = E(X_i) = 1
\]
\[
St. Dev. (\bar{X}) = \sqrt{Var(\bar{X})} = \sqrt{\frac{Var(X_i)}{100}} = \frac{St. Dev. (X_i)}{10} = 0.005
\]

Then, by the central limit theorem, we know that (approximately)

\[
\bar{X} \sim N \left( 1, (0.005)^2 \right)
\]

Therefore,

\[
\Pr (\bar{X} \geq 1.004) = \Pr \left( Z \geq \frac{1.004 - 1}{0.005} \right)
\]
\[
= \Pr (Z \geq 0.8)
\]
\[
\approx 0.2119
\]

where \(Z\) is a standard normal random variable.