Fallacies with Infinity

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Consider the following claim:

**Claim 1.** There exists an infinite decreasing sequence of natural numbers.

*Proof.* Assume for sake of contradiction that the longest decreasing sequence of natural numbers is finite. Let $S = \{a_1, a_2, \ldots, a_n\}$ be such a sequence. Then, choose some $a_0$ larger than $\max, a_i$, and note that $S' = \{a_0, a_1, a_2, \ldots, a_n\}$ forms a length-$(n + 1)$ decreasing sequence. This contradicts the maximality of $S$, and hence completes the proof. \(\square\)

Clearly, there must be something wrong with this proof, since there is no infinite decreasing sequence of natural numbers! What is the error?

The error lies in the first sentence of the proof: one is not allowed to assume that a longest decreasing sequence exists. In fact, in this case, a longest decreasing sequence does not exist. The Well Ordering Principle only allows us to assume that a shortest decreasing sequence exists.

A few students attempted proofs similar to this one for parts (a) and (b) of problem 3 of Problem Set 3. These solutions were not valid.

For completeness, let us look at another common error with infinity. We give another “proof” of Claim 1.

*Proof.* Let $P(n)$ be the statement, “There exists a decreasing length-$n$ sequence of natural numbers.”

We prove that $P(n)$ holds for all $n$ by induction. The base case $P(1)$, is clear, as we can just take the single-element sequence $\{1\}$. For the inductive step, suppose that $P(n)$ holds, and let $\{a_1, a_2, \ldots, a_n\}$ be a corresponding decreasing sequence. Then, choose some $a_0$ larger than $\max, a_i$, and note that $\{a_0, a_1, a_2, \ldots, a_n\}$ forms a length-$(n + 1)$ decreasing sequence. This demonstrates $P(n + 1)$ and hence completes the proof. \(\square\)

What is wrong with this proof? This proof demonstrates the common error of “infinite induction”. It is tempting to say that if $P(1)$ holds and if $P(n) \rightarrow P(n + 1)$ for all $n \in \mathbb{N}$, then $P(\infty)$ holds. However, the principle of induction does not guarantee this. It only guarantees that $P(n)$ holds for all $n \in \mathbb{N}$. In fact, the statement $P(\infty)$ is not always well-defined!

**Moral of the story:** Be very careful when dealing with infinity! When in doubt, run the proof by a TA or by your friend that has taken 18.100B.