Problem 1. (60 points) In the following, $X$ is a nonempty convex subset of $\mathbb{R}^n$, $A$ is a matrix of appropriate dimension, $b$ is a vector of appropriate dimension, and $f : \mathbb{R}^n \to (-\infty, \infty]$ is a convex proper function. State which of the following statements are true and which are false. You don’t have to justify your answers.

1. If the epigraph of $f$ is closed, $f$ is continuous.
2. If the epigraph of $f$ is closed, $\text{dom}(f)$ is closed.
3. The relative interior of $X$ is equal to its interior.
4. The recession cone of $X$ is equal to the recession cone of its relative interior.
5. There exists a hyperplane that separates $X$ and $-X$.
6. Let $f$ be the two-dimensional function $f(x_1, x_2) = (x_1 + x_2)^2$. Then $f$ is coercive.
7. If $f$ is closed and $\text{dom}(f)$ is compact then its conjugate is real-valued.
8. Suppose that the problem of minimizing $f$ over $x \in X$ and $Ax = b$ has finite optimal value and $X$ is open. Then there is no duality gap.
9. Suppose $f$ is the sum of a real-valued function and the indicator function of $X$. Then at each $x \in X$ there is at least one subgradient of $f$.
10. Suppose $f(x) = b^T x$ and $x^*$ minimizes $f$ over $X$. Then the normal cone of $X$ at $x^*$ contains $-b$.

Solution.

1. False. The given conditions only ensure that $f$ is lower semicontinuous.
2. False. Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = 1/x$.
3. False. Consider the interval $[-1, 1]$ on an axis of $\mathbb{R}^2$, which has empty interior but nonempty relative interior $(-1, 1)$.
4. False. Consider $X = \{x_1 > 0, x_2 \geq 0\} \cup \{(0, 0)\}$. Then the recession cone of $X$ is $X$ itself, while the recession cone of the relative interior of $X$ is the positive orthant.
5. False. If 0 is an interior point of $X$, then $X$ and $-X$ cannot be separated.
6. False. Along the line $x_1 + x_2 = 0$, even if $\sqrt{x_1^2 + x_2^2} \to \infty$, we have $f(x_1, x_2) = 0$.
7. True.
8. True. By the strong duality theorem.
9. True. Subgradients of the real-valued function and the indicator function of $X$ exist at all $x \in X$, and the relative interiors of their domains intersect. Use Prop. 5.4.6.
10. True.
Problem 2. (40 points) Consider the problem

$$\min x^2 + y^2$$

subject to \(a - x - y \leq 0, \ x, y \in \{0, 1\}\).

(a) Sketch the set of constraint-cost pairs:

$$\{(a - x - y, x^2 + y^2) : x, y \in \{0, 1\}\},$$

and the perturbation function

$$p(u) = \min_{a-x-y \leq u, \ x, y \in \{0, 1\}} x^2 + y^2.$$

Is \(p\) lower semicontinuous?

(b) Consider the MC/MC framework with \(M\) being the epigraph of \(p\). What are the values of \(a\) for which the problem is feasible and at the same time there is a duality gap? What are the values of \(a\) for which the problem is feasible and there is no duality gap? What are the values of \(a\) for which the problem is feasible and has a unique dual solution?

(c) Formulate the max crossing problem for one of the values of \(a\) for which the problem is feasible and there is no duality gap, and find the set of primal and dual optimal solutions.

(d) Replace the constraint \(a - x - y \leq 0\) with a strict inequality \(a - x - y < 0\). Answer the questions in parts (a) and (b) again.

Solution. (a) To be added. The perturbation function is

$$p(u) = \begin{cases} 
0 & \text{if } u \geq a, \\
1 & \text{if } a > u \geq a - 1, \\
2 & \text{if } a - 1 > u \geq a - 2, \\
\infty & \text{if } a - 2 > u,
\end{cases}$$

We can show that \(p\) is lower semicontinuous by verifying the definition.

(b) For the problem to be feasible, we must have \(a \leq 2\); and for there is a duality gap, from the MC/MC we see that \(a > 0, a \neq 1\) and \(a \neq 2\). To sum up, we have \(a \in (0, 1) \cup (1, 2)\).

For the problem to be feasible with no duality gap, we must have \(a \in (-\infty, 0] \cup \{1, 2\}\).

For the problem to be feasible with a unique dual solution, we have \(a \in (-\infty, 0) \cup (0, 2)\).

(c) Let \(a = 1\). The optimal solution is \((x^*, y^*) = (0, 1)\) or \((x^*, y^*) = (1, 0)\) and the optimal value is \(f^* = 1\). The max crossing problem is

$$\max_{\mu \geq 0} \inf_{u \in \mathbb{R}} \{p(u) + \mu' u\},$$

and the solution is \(q^* = 1\) and \(\mu^* = 1\).

(d) The function \(p\) is no longer semi-continuous. For the problem to be feasible with a duality gap, we must have \(a \in [0, 2)\). For the problem to be feasible with no duality gap, we must have \(a \in (-\infty, 0)\). For the problem to be feasible and has a unique dual solution, we have \(a \in (-\infty, 0) \cup (0, 2)\).
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