Class 3
Today’s plan

• Algorithms in general synchronous networks (continued):
  – Shortest paths spanning tree
  – Minimum-weight spanning tree
  – Maximal independent set

• Reading: Sections 4.3-4.5

• Next:
  – Distributed consensus
  – Reading: Sections 5.1, 6.1-6.3
Last time

- Lower bound on number of messages for comparison-based leader election in a ring.
- Leader election in general synchronous networks:
  - Flooding algorithm
  - Reducing message complexity
  - Simulation relation proof
- Breadth-first search in general synchronous networks:
  - Marking algorithm
  - Applications:
    - Broadcast, convergecast
    - Data aggregation (computation in networks)
    - Leader election in unknown networks
    - Determining the diameter
Termination for BFS

- Suppose $i_0$ wants to know when the BFS tree is completed.
- Assume each search message receives a response, parent or non-parent.
  - Easy if edges are bidirectional, harder if unidirectional.
- After a node has received responses to all its search messages, it knows who its children are, and knows they are all marked.
- Leaves of the tree discover who they are (receive all non-parent responses).
- Starting from the leaves, fan in complete messages to $i_0$.
- Node can send complete message after:
  - It has receives responses to all its search messages (so it knows who its children are), and
  - It has received complete messages from all its children.
Shortest paths

• **Motivation:** Establish structure for efficient communication.
  – Generalization of Breadth-First Search.
  – Now edges have associated costs (weights).

• **Assume:**
  – Strongly connected digraph, root $i_0$.
  – Weights (nonnegative reals) on edges.
    • Weights represent some communication cost, e.g. latency.
  – UIDs.
  – Nodes know weights of incident edges.
  – Nodes know $n$ (need for termination).

• **Required:**
  – Shortest-paths tree, giving shortest paths from $i_0$ to every other node.
  – Shortest path = path with minimum total weight.
  – Each node should output parent, “distance” from root (by weight).
Shortest paths
Shortest paths
Shortest paths algorithm

- **Bellman-Ford** (adapted from sequential algorithm)
- “Relaxation algorithm”
- Each node maintains:
  - dist, shortest distance it knows about so far, from i₀
  - parent, its parent in some path with total weight = dist
  - round number
- Initially i₀ has dist 0, all others ∞; parents all null
- At each round, each node:
  - Send dist to all out-nbrs
  - Relaxation step:
    - Compute new dist = min(dist, min_j(d_j + w_{ji})).
    - Update parent if dist changes.
- Stop after n-1 rounds
- Then (claim) dist contains shortest distance, parent contains parent in a shortest-paths tree.
Shortest paths

Round 1 (start)
Shortest paths

Round 1 (msgs)
Shortest paths

Round 1 (trans)
Shortest paths

Round 2 (start)
Shortest paths

Round 2 (msgs)
Shortest paths

Round 2 (trans)
Shortest paths

Round 3 (start)
Shortest paths

Round 3 (msgs)
Shortest paths

Round 3 (trans)
Shortest paths

Round 4 (start)
Shortest paths

Round 4 (msgs)
Shortest paths

Round 4 (trans)
Shortest paths
Shortest paths

Round 5 (trans)
Shortest paths

End configuration
Correctness

- Need to show that, after round n-1, for each process i:
  - \( \text{dist}_i \) = shortest distance from \( i_0 \)
  - \( \text{parent}_i \) = predecessor on shortest path from \( i_0 \)

- Proof:
  - Induction on the number \( r \) of rounds.
  - But, what statement should we prove about the situation after \( r \) rounds?
Correctness

- **Key invariant:** After $r$ rounds:
  - Every process $i$ has its $\text{dist}$ and $\text{parent}$ corresponding to a shortest path from $i_0$ to $i$ among those paths that consist of at most $r$ hops (edges).
  - If there is no such path, then $\text{dist} = \infty$ and $\text{parent} = \text{null}$.

- **Proof (sketch):**
  - By induction on the number $r$ of rounds.
  - **Base:** $r = 0$: Immediate from initializations.
  - **Inductive step:** Assume for $r-1$, show for $r$.
    - Fix $i$; must show that, after round $r$, $\text{dist}_i$ and $\text{parent}_i$ correspond to a shortest at-most-$r$-hop path.
    - First, show that, if $\text{dist}_i$ is finite, then it really is the distance on some at-most-$r$-hop path to $i$, and $\text{parent}$ is its parent on such a path.
    - LTTR---easy use of inductive hypothesis.
    - But we must still argue that $\text{dist}_i$ and $\text{parent}_i$ correspond to a shortest at-most-$r$-hop path.
Correctness

- **Key invariant:** After \( r \) rounds:
  - Every process \( i \) has its \( \text{dist} \) and \( \text{parent} \) corresponding to a shortest path from \( i_0 \) to \( i \) among those paths that consist of at most \( r \) hops (edges).
  - If there is no such path, then \( \text{dist} = \infty \) and \( \text{parent} = \text{null} \).

- **Proof, inductive step:**
  - Assume for \( r-1 \), show for \( r \).
  - Fix \( i \); must show that, after round \( r \), \( \text{dist}_i \) and \( \text{parent}_i \) correspond to a shortest at-most-\( r \)-hop path.
  - If \( \text{dist}_i \) is finite, then it really is the distance on some at-most-\( r \)-hop path to \( i \), and \( \text{parent} \) is its parent on such a path.
  - Claim that \( \text{dist}_i \) and \( \text{parent}_i \) correspond to a shortest at-most-\( r \)-hop path.
  - Any shortest at-most-\( r \)-hop path from \( i_0 \) to \( i \), when cut off at \( i \)'s predecessor \( j \) on the path, yields a shortest (\( r-1 \))-hop path from \( i_0 \) to \( j \).
  - By inductive hypothesis, after round \( r-1 \), for every such \( j \), \( \text{dist}_j \) and \( \text{parent}_j \) correspond to a shortest at-most-(\( r-1 \))-hop path from \( i_0 \) to \( j \).
  - At round \( r \), all such \( j \) send \( i \) their info about their shortest at-most-(\( r-1 \))-hop paths, and process \( i \) takes this into account in calculating \( \text{dist}_i \).
  - So after round \( r \), \( \text{dist}_i \) and \( \text{parent}_i \) correspond to a shortest at-most-\( r \)-hop path.
Complexity

- Complexity:
  - Time: $n-1$ rounds
  - Messages: $(n-1) |E|$

- Worse that BFS, which has:
  - Time: $diam$ rounds
  - Messages: $|E|$

- Q: Does the time bound really depend on $n$, or is it $O(diam)$?
- A: It’s really $n$, since “shortest path” can be over a path with more links.
- Example:
Bellman-Ford Shortest-Paths Algorithm

- Will revisit Bellman-Ford shortly in asynchronous networks.
- Gets even more expensive there.
- Similar to old Arpanet routing algorithm.
Minimum spanning tree

- Another classical problem.
- Many sequential algorithms.
- Construct a spanning tree, minimizing the total weight of all edges in the tree.

Assume:
- Weighted undirected graph (bidirectional communication).
  - Weights are nonnegative reals.
  - Each node knows weights of incident edges.
- Processes have UIDs.
- Nodes know (a good upper bound on) n.

Required:
- Each process should decide which of its incident edges are in MST and which are not.
Minimum spanning tree theory

• Graph theory definitions (for undirected graphs)
  – Tree: Connected acyclic graph
  – Forest: An acyclic graph (not necessarily connected)
  – Spanning subgraph of a graph G: Subgraph that includes all nodes of G.
    • Spanning tree, spanning forest.
  – Component of a graph: A maximal connected subgraph.

• Common strategy for computing MST:
  – Start with trivial spanning forest, n isolated nodes.
  – Repeat (n-1 times):
    • Merge two components along an edge that connects them.
    • Specifically, add the minimum-weight outgoing edge (MWOE) of some component to the edge set of the current forest.
Why this works:

- Similar argument to sequential case.
- **Lemma 1:** Let \{ \(T_i\) : \(1 \leq i \leq k\) \} be a spanning forest of \(G\). Fix any \(j, 1 \leq j \leq k\). Let \(e\) be a minimum weight outgoing edge of \(T_j\). Then there is a spanning tree for \(G\) that includes all the \(T_i\)'s and \(e\), and has minimum weight among all spanning trees for \(G\) that include all the \(T_i\)'s.
- **Proof:**
  - Suppose not---there’s some spanning tree \(T\) for \(G\) that includes all the \(T_i\)'s and does not include \(e\), and whose total weight is strictly less than that of any spanning tree that includes all the \(T_i\)'s and \(e\).
  - Construct a new graph \(T'\) (not a tree) by adding \(e\) to \(T\).
  - Contains a cycle, which must contain another outgoing edge, \(e'\), of \(T_j\).
  - \(\text{weight}(e') \geq \text{weight}(e)\), by choice of \(e\) (smallest weight).
  - Construct a new tree \(T''\) by removing \(e'\) from \(T'\).
  - Then \(T''\) is a spanning tree, contains all the \(T_i\)'s and \(e\).
  - \(\text{weight}(T'') \leq \text{weight}(T)\).
  - Contradicts assumed properties of \(T\).
Minimum spanning tree algorithms

• General strategy:
  – Start with n isolated nodes.
  – Repeat (n-1 times):
    • Choose some component i.
    • Add the minimum-weight outgoing edge (MWOE) of component i.

• Sequential MST algorithms follow (special cases of) this strategy:
  – **Dijkstra/Prim**: Grows one big component by adding one more node at each step.
  – **Kruskal**: Always add min weight edge globally.

• Distributed?
  – All components can choose simultaneously.
  – But there is a problem…
Can get cycles:
Avoid this problem by assuming that all weights are distinct.
Not a serious restriction---could break ties with UIDs.

**Lemma 2**: If all weights are distinct, then the MST is unique.
**Proof**: Another cycle argument (LTTR).

Justifies the following **concurrent strategy**:
- At each stage, suppose (inductively) that the current forest contains only edges from the unique MST.
- Now several components choose MWOEs concurrently.
- Each of these edges is in the unique MST, by Lemma 1.
- So OK to add them all (no cycles, since all are in the same MST).

**GHS (Gallager, Humblet, Spira) algorithm**
- Very influential (Dijkstra prize).
- Designed for asynchronous setting, but simplified here.
- We will revisit it in asynchronous networks.
GHS distributed MST algorithm

- Proceeds in *phases (levels)*, each with $O(n)$ rounds.
  - Length of phases is fixed, and known to everyone.
  - This is all that $n$ is used for.
  - We’ll remove use of $n$ for asynchronous algorithm.

- For each $k \geq 0$, level $k$ components form a spanning forest that is a subgraph of the unique MST.
- Each component is a tree rooted at a leader node.
  - Component identified by UID of leader.
  - Nodes in the component know which incident edges are in the tree.
- Each level $k$ component has at least $2^k$ nodes.
- Every level $k+1$ component is constructed from two or more level $k$ components.

- Level 0 components: Single nodes.
- Level $k \rightarrow$ level $k+1$: 
Level $k \rightarrow$ Level $k+1$

• Each level-$k$ component leader finds MWOE of its component:
  – Broadcasts search (via tree edges).
  – Each process finds the mwoe among its own incident edges.
    • Sends test messages along non-tree edges, asking if node at the other end is in the same component (compare component ids).
  – Convergecast the min back to the leader (via tree edges).
  – Leader determines MWOE.

• Combine level-$k$ components using MWOE$s$, to obtain level-$(k+1)$ components:
  – Wait long enough for all components to find MWOE$s$.
  – Leader of each level $k$ component tells endpoint nodes of its MWOE to add the edge for level $k+1$.
  – Each new component has $\geq 2^{k+1}$ nodes, as claimed.
Level k $\rightarrow$ Level k+1, cont’d

- Each level-k component leader finds MWOE of its component.
- Combine level-k components using MWOE, to obtain level-(k+1) components.

- Choose new leaders:
  - For each new, level k+1 component, there is a unique edge e that is the MWOE of two level k sub-components:
    - Choose new leader to be the endpoint of e with the larger UID.
    - Broadcast leader UID to new (merged) component.

- GHS terminates when there are no more outgoing edges.
Note on synchronization

• This simplified version of GHS is designed to work with component levels synchronized.
• Difficulties can arise when they get out of synch (as we’ll see).
• In particular, test messages are supposed to compare leader UIDs to determine whether endpoints are in the same component.
• Requires that the node being queried has up-to-date UID information.
Minimum spanning tree
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Simplified GHS MST Algorithm

- **Proof?**
- **Use invariants; but this is complicated because the algorithm is complicated.**
- **Complexity:**
  - **Time:** $O(n \log n)$
    - $n$ rounds for each level
    - $\log n$ levels, because there are $\geq 2^k$ nodes in each level $k$ component.
  - **Messages:** $O((n + |E|) \log n)$
    - Naïve analysis.
    - At each level, $O(n)$ messages sent on tree edges, $O(|E|)$ messages overall for all the test messages and their responses.
  - **Messages:** $O(n \log n + |E|)$
    - A surprising, significant reduction.
    - Trick also works in asynchronous setting.
    - Has implications for other problems, such as leader election.
O(n log n + |E|) message complexity

- Each process marks its incident edges as rejected when they are discovered to lead to the same component; no need to retest them.
- At each level, tests candidate edges one at a time, in order of increasing weight, until the first one is found that leads outside (or exhaust candidates).
- Rejects all edges that are found to lead to the same component.
- At next level, resumes where it left off.

- O(n log n + |E|) bound:
  - O(n) for messages on tree edges at each phase, O(n log n) total.
  - Test, accept (different component), reject (same component):
    - Amortized analysis.
    - Test-reject: Each (directed) edge has at most one test-reject, for O(|E|) total.
    - Test-accept: Can accept the same directed edge several times; but at most one test-accept per node per level, O(n log n) total.
Where/how did we use synchrony?

- Leader election
- Breadth-first search
- Shortest paths
- Minimum spanning tree

We will see these algorithms again in the asynchronous setting.
Spanning tree → Leader

• Given any spanning tree of an undirected graph, elect a leader:
  – Convergecast from the leaves, until messages meet at a node (which can become the leader) or cross on an edge (choose endpoint with the larger UID).
  – Complexity: Time $O(n)$; Messages $O(n)$

• Given any weighted connected undirected graph, with known $n$, but no leader, elect a leader:
  – First use GHS MST to get a spanning tree, then use the spanning tree to elect a leader.
  – Complexity: Time $O(n \log n)$; Messages $O(n \log n + |E|)$.
  – Example: In a ring, $O(n \log n)$ time and messages.
Other graph problems…

• We can define a distributed version of practically any graph problem: maximal independent set (MIS), dominating set, graph coloring,…
• Most of these have been well studied.
• For example…
Maximal Independent Set

- Subset I of vertices V of undirected graph G = (V,E) is independent if no two G-neighbors are in V.
- Independent set I is maximal if no strict superset of I is independent.
- Distributed MIS problem:
  - Assume: No UIDs, nodes know (good upper bound on) n.
  - Required:
    - Compute an MIS I of the network graph.
    - Each process in I should output winner, others output loser.
- Application: Wireless network transmission
  - A transmitted message reaches neighbors in the graph; they receive the message if they are in “receive mode”.
  - Let nodes in the MIS transmit messages simultaneously, others receive.
  - Independence guarantees that all transmitted messages are received by all neighbors (since neighbors don’t transmit at the same time).
  - Neglecting collisions here---some strategy (backoff and retransmission, or coding) is needed for this.
- Unsolvable by deterministic algorithm, in some graphs.
- Randomized algorithm [Luby]:
Luby’s MIS Algorithm (sketch)

- Each process chooses a random val in \{1,2,…,n^4\}.
  - Large enough set so it’s very likely that all numbers are distinct.
- Neighbors exchange vals.
- If node i’s val > all neighbors’ vals, then process i declares itself a winner and notifies its neighbors.
- Any neighbor of a winner declares itself a loser, notifies its neighbors.
- Processes reconstruct the remaining graph, eliminating winners, losers, and edges incident on winners and losers.
- Repeat on the remaining graph, until no nodes are left.

**Theorem:** If LubyMIS ever terminates, it produces an MIS.

**Theorem:** With probability 1, it eventually terminates; the expected number of rounds until termination is O(log n).

**Proof:** LTTR.
Termination theorem for Luby MIS

- **Theorem:** With probability 1, Luby MIS eventually terminates; the expected number of rounds until termination is $O(\log n)$.

- **Proof:** Key ideas
  - Define $\text{sum}(i) = \sum_{j \in \text{nbrs}(i)} 1/\text{degree}(j)$.
    - Sum of the inverses of the neighbors’ degrees.
  - **Lemma 1:** In one stage of Luby MIS, for each $i$ in the graph, the probability that $i$ is a loser (neighbor of a winner) is $\geq 1/8 \cdot \text{sum}(i)$.
  - **Lemma 2:** The expected number of edges removed from $G$ in one stage is $\geq |E| / 8$.
  - **Lemma 3:** With probability at least $1/16$, the number of edges removed from $G$ at a single stage is $\geq |E| / 16$. 
Next time

- Distributed consensus
- Reading: Sections 5.1, 6.1-6.3