A set $A$ of natural numbers is said to be \textit{m-reducible} (for "many-one reducible") to a set $B$ just in case there is a total $\Sigma$ function $f$ such that, for any $n$, $n$ is in $A$ if and only if $f(n)$ is in $B$. $A$ is \textit{I-reducible} (for "one-one reducible") to $B$ just in case there is a one-one total $\Sigma$ function $f$ such that, for any $n$, $n$ is in $A$ if and only if $f(n)$ is in $B$.

1. Show that the following are equivalent, for any set $A$:
   (i) $A$ is recursively enumerable (that is, $\Sigma$)
   (ii) $A$ is 1-reducible to the set of Gödel numbers of valid sentences
   (iii) $A$ is m-reducible to the set of Gödel numbers of valid sentences.

   (i) $\Rightarrow$ (ii). If $A$ is recursively enumerable, then there is a $\Sigma$ formula $\phi(x)$, with "$x$" as its only free variable, that weakly represents $A$ in $Q$. If we set $f(n)$ equal to $\neg (Q \rightarrow \phi([n]))\psi$ (where "$Q$" denotes the conjunction of the axioms of Robinson's arithmetic), then we have $n \in A$ iff $Q \rightarrow \phi([n])$ iff $(Q \rightarrow \phi([n]))$ is valid iff $f(n) \in \{\text{Gödel numbers of valid sentences}\}$.

   (ii) $\Rightarrow$ (iii). Trivial.

   (iii) $\Rightarrow$ (i). Take a one-one total $\Sigma$ function $f$ such that, for any $n$, we have $n \in A$ iff $f(n) \in \{\text{Gödel numbers of valid formulas}\}$; we can find a bounded formula $\phi(x,y,z)$ such that, for any $n$ and $m$, we have $f(n) = m$ iff $(\exists z)\phi([n],[m],z)$ is true. We know that the set of Gödel numbers of valid formulas is $\Sigma$, so that there is a bounded formula $\psi(x,y)$ such that, for any $m$, $m$ is the Gödel number of a valid sentence iff $(\exists y)\psi([m],y)$ is true. Then, for any $n$, $n \in A$ iff the $\Sigma$ formula $(\exists y)(\exists z)(\exists w)(\phi([n],y,z) \land \psi(y,w))$ is true.

2. Give an example of a $\Sigma$ partial function that cannot be extended to a $\Sigma$ total function.

   Let $f$ be the partial function that gives the value 1 if the input is (the Gödel number of) a theorem of $Q$, the value 0 if the input is a sentence refutable in $Q$, and is undefined otherwise. $f$ is a $\Sigma$ partial function, but it cannot be extended to a $\Sigma$ total function, since if $g$ were such a function, the $\Sigma$ total function that takes $x$ to $\max(g(x), 1)$ would be the characteristic function of a recursive set that separates the theorems of $Q$ from the sentences refutable in $Q$. 