Lecture 11: Max/Min Problems

**Example 1.** \( y = \frac{\ln x}{x} \) (same function as in last lecture)

![Graph of \( y = \frac{\ln x}{x} \).](image)

Figure 1: Graph of \( y = \frac{\ln x}{x} \).

- What is the maximum value? Answer: \( y = \frac{1}{e} \).
- Where (or at what point) is the maximum achieved? Answer: \( x = e \). (See Fig. 1.)

**Beware:** Some people will ask “What is the maximum?” The answer is *not* \( e \). You will get so used to finding the critical point \( x = e \), the main calculus step, that you will forget to find the maximum value \( y = \frac{1}{e} \). Both the critical point \( x = e \) and critical value \( y = \frac{1}{e} \) are important. Together, they form the point of the graph \((e, \frac{1}{e})\) where it turns around.

**Example 2.** Find the max and the min of the function in Fig. 2

Answer: If you’ve already graphed the function, it’s obvious where the maximum and minimum values are. The point is to find the maximum and minimum without sketching the whole graph.

**Idea:** Look for the max and min among the critical points and endpoints. You can see from Fig. 2 that we only need to compare the heights or \( y \)-values corresponding to endpoints and critical points. (Watch out for discontinuities!)
Figure 2: Search for max and min among critical points and endpoints

**Example 3.** Find the open-topped can with the least surface area enclosing a fixed volume, $V$.

1. Draw the picture.
2. Figure out what variables to use. (In this case, $r$, $h$, $V$ and surface area, $S$.)
3. Figure out what the constraints are in the problem, and express them using a formula. In this example, the constraint is

$$V = \pi r^2 h = \text{constant}$$

We’re also looking for the surface area. So we need the formula for that, too:

$$S = \pi r^2 + (2\pi r)h$$

Now, in symbols, the problem is to minimize $S$ with $V$ constant.
4. Use the constraint equation to express everything in terms of $r$ (and the constant $V$).

\[ h = \frac{V}{2\pi r}; \quad S = \pi r^2 + (2\pi r) \left( \frac{V}{\pi r^2} \right) \]

5. Find the critical points (solve $dS/dr = 0$), as well as the endpoints. $S$ will achieve its max and min at one of these places.

\[ \frac{dS}{dr} = 2\pi r - \frac{2V}{r^2} = 0 \implies \pi r^3 - V = 0 \implies r^3 = \frac{V}{\pi} \implies r = \left( \frac{V}{\pi} \right)^{1/3} \]

We’re not done yet. We’ve still got to evaluate $S$ at the endpoints: $r = 0$ and “$r = \infty$”.

\[ S = \pi r^2 + \frac{2V}{r}, \quad 0 \leq r < \infty \]

As $r \to 0$, the second term, $\frac{2V}{r}$, goes to infinity, so $S \to \infty$. As $r \to \infty$, the first term $\pi r^2$ goes to infinity, so $S \to \infty$. Since $S = +\infty$ at each end, the minimum is achieved at the critical point $r = (V/\pi)^{1/3}$, not at either endpoint.

![Figure 4: Graph of $S$](image)

We’re still not done. We want to find the minimum value of the surface area, $S$, and the values of $h$.

\[ r = \left( \frac{V}{\pi} \right)^{1/3}; \quad h = \frac{V}{\pi r^2} = \frac{V}{\pi} \left( \frac{V}{\pi} \right)^{2/3} = \frac{V}{\pi} \left( \frac{V}{\pi} \right)^{-2/3} = \left( \frac{V}{\pi} \right)^{1/3} \]

\[ S = \pi r^2 + \frac{2V}{r} = \pi \left( \frac{V}{\pi} \right)^{2/3} + 2V \left( \frac{V}{\pi} \right)^{1/3} = 3\pi^{-1/3}V^{2/3} \]

Finally, another, often better, way of answering that question is to find the proportions of the can. In other words, what is $\frac{h}{r}$? Answer: $\frac{h}{r} = \left( \frac{V}{\pi} \right)^{1/3} = 1$. 

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**Example 4.** Consider a wire of length 1, cut into two pieces. Bend each piece into a square. We want to figure out where to cut the wire in order to enclose as much area in the two squares as possible.

![Diagram](image.png)

The first square will have sides of length $\frac{x}{4}$. Its area will be $\frac{x^2}{16}$. The second square will have sides of length $\frac{1-x}{4}$. Its area will be $\left(\frac{1-x}{4}\right)^2$. The total area is then

\[
A = \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2
\]

\[
A' = \frac{2x}{16} + \frac{2(1-x)}{16}(-1) = \frac{x}{8} - \frac{1}{8} + \frac{x}{8} = 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2}
\]

So, one extreme value of the area is

\[
A = \left(\frac{\frac{1}{2}}{4}\right)^2 + \left(\frac{\frac{1}{2}}{4}\right)^2 = \frac{1}{32}
\]

We’re not done yet, though. We still need to check the endpoints! At $x = 0$,

\[
A = 0^2 + \left(\frac{1-0}{4}\right)^2 = \frac{1}{16}
\]

At $x = 1$,

\[
A = \left(\frac{\frac{1}{4}}{4}\right)^2 + 0^2 = \frac{1}{16}
\]
By checking the endpoints in Fig. 6, we see that the minimum area was achieved at $x = \frac{1}{2}$. The maximum area is not achieved in $0 < x < 1$, but it is achieved at $x = 0$ or $1$. The maximum corresponds to using the whole length of wire for one square.

![Figure 6: Graph of the area function.](image)

**Moral**: Don’t forget endpoints. If you only look at critical points you may find the worst answer, rather than the best one.