Lecture 21: The mean value inequality for uniformly elliptic operators part I

1 The mean value inequality: Iterative argument

In this lecture we will prove a mean value inequality for uniformly elliptic operators in divergence form. The argument is an iterative one due to De Georgi, Nash, and Moser. As usual we take \( L \) an operator with

\[ Lu = \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} \] (1)

and \( \lambda |v|^2 \leq A_{ij} v_i v_j \leq \Lambda |v|^2 \) for all vectors \( v \). Let \( u \) be a function satisfying \( u \geq 0, Lu \geq 0 \). Take \( x_0 \) a point, and \( R \) a fixed positive number. Let \( \phi \) be a test function on \( B_R(x_0) \) which is zero on the boundary. Clearly

\[ \int_{B_R(x_0)} \phi^2 u A \nabla u \cdot dS = 0 \] (2)

so, by Stokes’ theorem,

\[ \int_{B_R(x_0)} \phi^2 u Lu + \int_{B_R(x_0)} A_{ij} \frac{\partial \phi^2 u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0 \] (3)

and, since the first term is non-negative,

\[ 0 \geq \int_{B_R(x_0)} A_{ij} \frac{\partial \phi^2 u}{\partial x_i} \frac{\partial u}{\partial x_j}. \] (4)

We can simplify this a bit to get

\[ 0 \geq \int_{B_R(x_0)} A_{ij} \phi^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2 \int_{B_R(x_0)} A_{ij} \phi u \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j}. \] (5)

and

\[ -2 \int_{B_R(x_0)} A_{ij} \phi u \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j} \geq \int_{B_R(x_0)} A_{ij} \phi^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}. \] (6)
Apply uniform ellipticity to the right hand side to get
\[ \lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq -2 \int_{B_R(x_0)} A_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j}. \]  
(7)

Now work on the other term. At each point the matrix \( A \) defines a good metric, so Cauchy-Schwarz applies, and we get
\[-\phi u A_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j} \leq \phi u (\nabla \phi \cdot A \nabla \phi)^{1/2} (\nabla u \cdot A \nabla u)^{1/2},\]
so
\[ \lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq 2 \int_{B_R(x_0)} \phi u (\nabla \phi \cdot A \nabla \phi)^{1/2} (\nabla u \cdot A \nabla u)^{1/2}. \]  
(8)

Use Cauchy-Schwarz again in the form \( \int fg \leq (\int f^2)^{1/2} (\int g^2)^{1/2} \) to get
\[ \lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq 2 \left( \int_{B_R(x_0)} u^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \right)^{1/2}. \]  
(9)

Uniform ellipticity then gives
\[ \lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq 2 \Lambda \left( \int_{B_R(x_0)} u^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \right)^{1/2}, \]  
(10)
so rearrange to get
\[ \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq \frac{4 \Lambda^2}{\lambda^2} \int_{B_R(x_0)} u^2 |\nabla \phi|^2. \]  
(11)

This should be familiar, as we proved it on the way to the Cacciopoli inequality in lecture 6. We'll apply it slightly differently this time. Consider
\[ \int_{B_R(x_0)} |\nabla (\phi u)|^2 = \int_{B_R(x_0)} |\phi \nabla u + u \nabla \phi|^2 \]  
(12)
\[ \leq 2 \int_{B_R(x_0)} \phi^2 |\nabla u|^2 + 2 \int_{B_R(x_0)} u^2 |\nabla \phi|^2. \]  
(13)

Combining this with 11 we get
\[ \int_{B_R(x_0)} |\nabla (\phi u)|^2 \leq k \int_{B_R(x_0)} u^2 |\nabla \phi|^2 \]  
(14)
for a constant \( k = 2 + \frac{8 \Lambda^2}{\lambda^2} \). Now we need to use the Sobolev inequality. For simplicity we will assume that \( n \geq 3 \), but a similar result holds in the other cases.
Theorem 1.1 Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, and let $w$ be a function with compact support on $\Omega$. Then

\[
\left( \int_{\Omega} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c \int_{\Omega} |\nabla w|^2.
\]  

(15)

We won’t prove this here. Apply it with $w = \phi u$ (this has compact support because $\phi$ does) to get

\[
\int_{B_R(x_0)} (\phi u)^{\frac{2n}{n-2}} \leq c \int_{B_R(x_0)} |\nabla (\phi u)|^2 \leq \tilde{c} \int_{B_R(x_0)} u^2 |\nabla \phi|^2.
\]  

(16)

for some constant $\tilde{c}$.

Define $A_{r,k} = B_r(x_0) \cap \{ u > k \}$, and let $|A_{r,k}|$ be the volume of this set. For any function $f$ define $f_+$ to be the positive part, i.e.

\[
f_+ = \sup(f, 0).
\]  

(17)

Note that if $u$ is $L$ harmonic then $u_+$ is $L$ harmonic almost everywhere, and claim without proof that everything we’ve done today goes through for the positive part of a harmonic function as well as for completely harmonic functions. Also pick $r < R$, and set

\[
\phi = \begin{cases} 
1 & \text{on } B_r(x_0) \\
\frac{R - |x|}{R - r} & \text{on } B_R(x_0) \setminus B_r(x_0), \text{ and} \\
0 & \text{outside } B_R(x_0)
\end{cases}
\]  

(18)

so that $|\nabla \phi| = \frac{1}{R - r}$ on $B_R(x_0)$, and 0 elsewhere. Note that if $u$ is $L$-harmonic then $u - k$ is also $L$ harmonic. Putting all this together we get

\[
\left( \int_{A_{r,k}} (u - k)_+^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \left( \int_{B_R(x_0)} (\phi (u - k)_+)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}
\]  

(19)

\[
\leq \tilde{c} \int_{B_R(x_0)} |\nabla \phi|^2 ((u - k)_+)^2
\]  

(20)

\[
\leq \frac{\tilde{c}}{(R - r)^2} \int_{A_{r,k} \setminus B_r(x_0)} ((u - k)_+)^2.
\]  

(21)

Now we’ll introduce another important inequality: the Holder Inequality.

Theorem 1.2 Let $f, g$ be functions, and $p, q$ real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then

\[
\int fg \leq \left( \int f^p \right)^{1/p} \left( \int g^q \right)^{1/q}.
\]  

(22)
This is simply a generalisation of the Cauchy-Schwarz inequality, which is the case $p = q = 2$. Apply this with $p = \frac{n}{n-2}$, $q = \frac{n}{2}$ and any function $f$ on any set $\Omega$ to get
\[ \int_{\Omega} f^2 \leq \left( \int_{\Omega} \left( \frac{f}{\sqrt{n}} \right)^{2n-2} \right)^{\frac{n-2}{n}} |\Omega|^\frac{2}{n}. \] (23)

Set $f = (u - k)_+$ and $\Omega = A_{r,k}$ and we get
\[ \int_{A_{r,k}} ((u - k)_+)^2 \leq \left( \int_{A_{r,k}} ((u - k)_+)^{2n-2} \right)^{\frac{n-2}{n}} |A_{r,k}|^\frac{2}{n} \] (24)
\[ \leq \frac{\tilde{c}|A_{r,k}|^{\frac{2}{n}}}{(R-r)^2} \int_{A_{r,k} \setminus B_r(x_0)} ((u - k)_+)^2 \] (25)
\[ \leq \frac{\tilde{c}|A_{r,k}|^{\frac{2}{n}}}{(R-r)^2} \int_{A_{r,k}} ((u - k)_+)^2. \] (26)

Note that if $h < k$ then $A_{r,k} \subset A_{r,h}$. Take $x \in A_{r,k}$, then $u(x) > k$, and $u(x) - h > k - h$. Therefore
\[ \int_{A_{r,k}} ((u - h)_+)^2 \geq \int_{A_{r,k}} (k - h)^2 = (k - h)^2 |A_{r,k}| \] (27)
and
\[ |A_{r,k}| \leq \frac{1}{(k - h)^2} \int_{A_{r,k}} ((u - h)_+)^2 \leq \frac{1}{(k - h)^2} \int_{A_{r,h}} ((u - h)_+)^2. \] (28)
for all $h < k$. Plugging this back into 26 we get
\[ \int_{A_{r,k}} ((u - k)_+)^2 \leq \frac{\tilde{c}}{(R-r)^2(k - h)^4/n} \left( \int_{A_{r,h}} ((u - h)_+)^2 \right)^{2/n} \int_{A_{r,k}} ((u - k)_+)^2 \] (29)
\[ \leq \frac{\tilde{c}}{(R-r)^2(k - h)^4/n} \left( \int_{A_{r,h}} ((u - h)_+)^2 \right)^{2/n} \int_{A_{r,h}} ((u - h)_+)^2 \] (30)
\[ \leq \frac{\tilde{c}}{(R-r)^2(k - h)^4/n} \left( \int_{A_{r,h}} ((u - h)_+)^2 \right)^{(1+2/n)} \] (31)

Next lecture we will actually do the induction argument.