Elliptic regularity

Hitherto we have always assumed our solutions already lie in the appropriate $C^{k,\alpha}$ space and then showed estimates on their norms in those spaces. Now we will avoid this a priori assumption and show that they do hold a posteriori. This is important for the consistency of our discussion. Precisely what we would like to show is —

**A priori regularity.** Let $u \in C^2(\Omega)$ be a solution of $Lu = f$ and assume $0 < \alpha < 1$. We do not assume $c(x) \leq 0$ but we do assume all the other assumptions on $L$ in the previous Theorem hold. If $f \in C^\alpha(\Omega)$ then $u \in C^{2,\alpha}(\Omega)$.

- Here we mean the $C^\alpha$ norm is locally bounded, i.e for every point exists a neighborhood where the $C^\alpha$-norm is bounded. Had we written $C^\alpha(\bar{\Omega})$ we would mean a global bound on $\sup_{x,y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$ (as in the footnote if Lecture 14).
- This result will allow us to assume in previous theorems only $C^2$ regularity on (candidate) solutions instead of assuming $C^{2,\alpha}$ regularity.

**Proof.** Let $u$ be a solution as above. Since the Theorem is local in nature we take any point in $\Omega$ and look at a ball $B$ centered there contained in $\Omega$. We then consider the Dirichlet problem

$$
\begin{align*}
L_0 v &= f' \quad \text{on} \quad B, \\
v &= u \quad \text{on} \quad \partial B.
\end{align*}
$$

where $L_0 := L - c(x)$ and $f'(x) := f(x) - c(x) \cdot u(x)$. This Dirichlet problem is on a ball, with "$c \leq 0$", uniform elliptic and with coefficients in $C^\alpha$. Therefore we have uniqueness and existence of a solution $v$ in $C^{2,\alpha}(B) \cap C^0(\bar{B})$. But $u$ satisfies $Lu = f$ or equivalently $L_0u = f'$ on all of $\Omega$ so
in particular on $\bar{B}$. By uniqueness on $B$ therefore we have $u|_{\bar{B}} = v$, and so $u$ is $C^{2,\alpha}$ smooth there. As this is for any point and all balls we have $u \in C^{2,\alpha}$. 

It is insightful to note at this point that these results are optimal under the above assumptions. Indeed need $C^2$ smoothness (or atleast $C^{1,1}$) in order to define $2^{\text{nd}}$ derivatives wrt $L$! If one takes $u$ in a larger function space, i.e weaker regularity of $u$, and defines $Lu = f$ in a weak sense then need more regularity on coefficients of $L$! Under the assumption of $C^\alpha$ continuity on the coefficients indeed we are in an optimal situation.

**Higher a priori regularity.** Let $u \in C^2(\Omega)$ be a solution of $Lu = f$ and $0 < \alpha < 1$. We do not assume $c(x) \leq 0$ but we assume uniformly elliptic and that all coefficients are in $C^{k,\alpha}$. If $f \in C^{k,\alpha}$ then $u \in C^{k+2,\alpha}$. If $f \in C^\infty$ then $u \in C^\infty$.

**Proof.** $k = 0$ was the previous Theorem.

**The case** $k = 1$. The proof relies in an elegant way on our previous results with the combination of the new idea of using difference quotients. We would like to differentiate the $u$ three times and prove we get a $C^\alpha$ function. Differentiating the equation $Lu = f$ once would serve our purpose but it can not be done na"ively as it would involve 3 derivatives of $u$ and we only know that $u$ has two. To circumvent this hurdle we will take two derivatives of the difference quotients of $u$, which we define by (let $e_1, \ldots, e_n$ denote the unit vectors in $\mathbb{R}^n$)

$$\Delta^h u := \frac{u(x + h \cdot e_1) - u(x)}{h} = u^h(x) - u(x).$$

Namely we look at

$$\Delta^h Lu = \frac{Lu(x + h \cdot e_1) - Lu(x)}{h} = \frac{f(x + h \cdot e_1) - f(x)}{h} = \Delta^h uf.$$

Note $\Delta^h v(x) \to Dlv(x)$ if $v \in C^1$ (which we don’t know a priori in our case yet).

Expanding our equation in full gives
\[
\frac{1}{h} \left[ (a^{ij}(x + h \cdot e_1) - a^{ij}(x))D_{ij}u^h - a^{ij}(x)D_{ij}u(x) + b^{i}(x + h \cdot e_1)D_iu(x + h \cdot e_1) - b^{i}(x)D_iu(x) + c(x + h \cdot e_1)u(x + h \cdot e_1) - c(x)u(x) \right]
\]

\[= \Delta^h a^{ij}D_{ij}u^h - a^{ij}D_{ij}\Delta^h u + \Delta^h b^iD_iu^h + b^iD_i\Delta^h u + \Delta^h c \cdot u^h + c \cdot \Delta^h u = \Delta^h f.\]

or succintly

\[L\Delta^h u = f' := \Delta^h f - \Delta^h a^{ij} \cdot D_{ij}u^h - \Delta^h b_i \cdot D_iu^h - \Delta^h c \cdot u^h\]

where \(u^h := u(x + h \cdot e_1)\).

We now analyse the regularity of the terms. \(f \in C^{1,\alpha}\) so so is \(\Delta^h f\), but not (bounded) uniformly \(\text{wrt } h\) (i.e \(C^{1,\alpha}\) norm of \(\Delta^h f\) may go to \(\infty\) as \(h\) decreases). On the otherhand \(\Delta^h f \in C^{\alpha}(\Omega)\) uniformly \(\text{wrt } h\) (\(\forall h > 0\)): \(\Delta^h u f = \frac{f(x + h \cdot e_1) - f(x)}{h} = D_i f(\bar{x})\) for some \(\bar{x}\) in the interval, and RHS has a uniform \(C^{\alpha}\) bound as \(f \in C^{1,\alpha}\) on all \(\Omega\) (needed as \(\bar{x}\) can be arbitrary).

For the same reason \(\Delta^h a^{ij}, \Delta^h b_i, \Delta^h c \in C^{\alpha}(\Omega)\). By the \(k = 0\) case we know \(u \in C^{2,\alpha}(\Omega)\) and not just in \(C^{2}(\Omega)\). \(\iff \) \(D_{ij}u^h \in C^{\alpha}(\Omega)\) uniformly.

\textbf{Remark.} We take a moment to describe what we mean by uniformity. We say a function \(g_h = g(h, \cdot) : \Omega \rightarrow \mathbb{R}\) is \textit{uniformly bounded in }\(C^{\alpha}\) \textit{wrt } \(h\) when \(\forall \Omega' \subseteq \Omega\) exists \(c(\Omega)\) such that \(|g_h|_{C^{\alpha}(\Omega')} \leq c(\Omega)\). Note this definition goes along with our \textit{local} definition of a function being in \(C^{\alpha}(\Omega)\) (and not in \(C^{\alpha}(\Omega)!\)).

Putting the above facts together we now see that both sides of the equation \(L\Delta^h u = f'\) are in \(C^{\alpha}(\Omega)\). And they are also in \(C^{\alpha}(\Omega')\) with RHS uniformly so with constant \(c(\Omega')\).

By the interior Schauder estimate, \(\forall \Omega'' \subseteq \Omega'\) and for each \(h\)

\[||\Delta^h u||_{C^{2,\alpha}(\Omega'')} \leq c(\gamma, \Lambda, \Omega'') \cdot (||\Delta^h u||_{C^{\alpha,\alpha'}(\Omega')} ||f'||_{C^{\alpha,\alpha'}(\Omega)}) \leq \tilde{c}(\gamma, \Lambda, \Omega'', \Omega', \Omega, ||u||_{C^{1}(\Omega)},\]

which is independent of \(h\)! If we assume the Claim below taking the limit \(h \rightarrow 0\) we get \(D_l u \in C^{2,\alpha}(\Omega''), \forall l = 1, \ldots, n\) \(u \in C^{3,\alpha}(\Omega''). \forall \Omega'' \subseteq \Omega' \subseteq \Omega \iff u \in C^{3,\alpha}(\Omega)\). \(\blacksquare\)
Claim. \[ \|\Delta^h g\|_{C^\alpha(A)} \leq c \text{ independently of } h \iff D_l g \in C^\alpha(A). \]

First we show \( g \in C^{0,1}(A) \). This is tantamount to the existence of \( \lim_{h \to 0} \Delta^h g(x) \) (since if it exists it equals \( D_l u \gamma(x) \) - that’s how we define the first \( l \)-directional derivative at \( x \)). Now \( \{\Delta^h g\}_{h>0} \) is family of uniformly bounded (in \( C^0(A) \)) and equicontinuous functions (from the uniform Hölder constant). So by the Arzelà-Ascoli Theorem exists a sequence \( \{\Delta^h g\}_{i=1}^\infty \) converging to some \( \tilde{w} \in C^\alpha(A) \) in the \( C^\beta(A) \) norm for any \( \beta < \alpha \). But as we remarked above \( \tilde{w} \) necessarily equals \( D_l g \) by definition.

Second, we show \( g \in C^1(A) \) (i.e such that derivative is continuous not just bounded) and actually \( \in C^{1,\alpha}(A) \):

\[
c \geq \|\Delta^h g\|_{C^\alpha(A)} \geq \lim_{h \to 0} \frac{\Delta^h g(x) - \Delta^h g(y)}{|x-y|^{\alpha}} = \frac{D_l g(x) - D_l g(y)}{|x-y|^{\alpha}} = |D_l g|_{C^\alpha(A)}
\]

where we used that \( c \) is independent of \( h \).

The case \( k \geq 2 \). Let \( k = 2 \). By the \( k = 1 \) case we can legitimately take 3 derivatives as \( u \in C^{3,\alpha}(\Omega) \). One has

\[
L(D_l u) = f' := D_l f - D_l a^{ij} \cdot D_{ij} u - D_l b_i \cdot D_i u - D_l c \cdot u
\]

with \( D_l u, f' \in C^{1,\alpha}(\Omega) \). So again by the \( k = 1 \) case we have now \( D_l u \in C^{3,\alpha}(\Omega) \), hence \( u \in C^{4,\alpha}(\Omega) \). The instances \( k \geq 3 \) are in the same spirit.

Boundary regularity

Let \( \Omega \) be a \( C^{2,\alpha} \) domain, i.e whose boundary is locally the graph of a \( C^{2,\alpha} \) function. Let \( L \) be uniformly elliptic with \( C^\alpha \) coefficients and \( c \leq 0 \).

Theorem. Let \( f \in C^\alpha(\Omega), \varphi \in C^{2,\alpha}(\partial \Omega), u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfying \( Lu = f \) on \( \Omega \),

\[
u = \varphi \quad \text{on} \quad \partial \Omega.
\]

with \( 0 < \alpha < 1 \). Then \( u \in C^{2,\alpha}(\overline{\Omega}) \).
Proof. Our previous results give \( u \in C^{2,\alpha}(\Omega) \) and we seek to extend it to those points in \( \partial \Omega \). Note that even though \( u = \varphi \) on \( \partial \Omega \) and \( \varphi \) is \( C^{2,\alpha} \) there this does not give the same property for \( u \). It just gives that \( u \) is \( C^{2,\alpha} \) in directions tangent to \( \partial \Omega \), but not in directions leading to the boundary.

The question is local: restrict attention to \( B(x_0, R) \cap \bar{\Omega} \) for each \( x_0 \in \partial \Omega \). We choose a \( C^{2,\alpha} \) homeomorphism \( \Psi_1 : \mathbb{R}^n \to \mathbb{R}^n \) sending \( B(x_0, R) \cap \partial \Omega \) to a portion of a (flat) hyperplane and \( \partial B(x_0, R) \cap \Omega \) to the boundary of half a disc. We then choose another \( C^{2,\alpha} \) homeomorphism \( \Psi_2 : \mathbb{R}^n \to \mathbb{R}^n \) sending the whole half disc into a disc (= a ball). Therefore \( \Psi_2 \circ \Psi_1 \) maps our original boundary portion into a portion of the boundary of a ball.

Similarly to previous computations of this sort we define the induced operator \( \tilde{L} \) on the induced domain \( \Psi_2 \circ \Psi_1(B(x_0, R) \cap \Omega) \) and define the induced functions \( \tilde{u}, \tilde{\varphi}, \tilde{f} \) and we get a new Dirichlet problem with all norms of our original objects equivalent to those of our induced ones. Note that still \( \tilde{c} := c \circ \Psi_1^{-1} \circ \Psi_2^{-1} \leq 0 \), therefore by our theory exists a unique solution \( v \in C^{2,\alpha}(\Psi_2 \circ \Psi_1(B(x_0, R) \cap \Omega) \cup \Psi_2 \circ \Psi_1(B(x_0, R) \cap \partial \Omega)) \cap C^0(\Psi_2 \circ \Psi_1(B(x_0, R) \cap \bar{\Omega})) \) for the induced Dirichlet problem. Now our \( \tilde{u} \) also solves it. So by uniqueness \( \tilde{u} = v \) and \( \tilde{u} \) has \( C^{2,\alpha} \) regularity as the induced boundary portion, and by pulling back through \( C^{2,\alpha} \) diffeomorphisms we get that so does \( u \).

Remark. The assumption \( c \leq 0 \) is not necessary although modifying the proof is non-trivial without this assumption (exercise). We needed it in order to be able to use our existence result. But since we already assume a solution exists we may use some of our previous results which do not need \( c \leq 0 \) and which secure \( C^{2,\alpha} \) regularity up to the boundary.